OPUSCULA
STATICO-MECHANICA
PRINCIPIIS
ANALYSEOS FINITORUM
SUPERSTRUCTA

EDITORE

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VOLUMEN I.

CUM FIGURIS.

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PRAEFATIO.

Amplitudo ſcientiarum ſtatico-mechanicarum eſt unum e praecipuis impedimentis, quae in illarum ſtudio progredi cupientibus iter jam retardant ac remorantur, jam penitus intercludunt. Tot enim inventis ſunt hae ſcientiae hactenus amplificatae, totque novis progreſſibus et acceſſionibus illae indies adaugentur et promoventur, ut, qui vel primos ordines in iis ducere ſibi propoſuerit, in latiſſime patentem campum debeat excurrere; tot opera diverſa et diaria conſulere, ut raro unus ſit futurus, qui illa conquirere poſſit et pervolvere. In hac porro inventorum ubertate multa occurrunt peculiari ſtudio exquirenda; multa ad criticae normam expendenda, et recto acutoque examine diiudicanda, quo minus utilia ab utilibus, falſa a veris, et incerta a certis probe diſcernantur.

 Indu-

Indubium eſt, partem puram ſcientiarum ſtatico-mechanicarum ad tantum perfectionis gradum jam eſſe perductam, ut fere nihil ſit, quod expectari poſſit aut debeat: ſeu enim aequilibrii, ſeu motus principia conſideres, ad tantam ſunt ea univerſalitatem promota, ut nihil amplius deſiderandum videatur. Interea certum eſt, pleraque principia a multis harum ſcientiarum doctoribus aut ex alienis et minus genuinis principiis eſſe derivata, aut methodo inconvenienti expoſita: apud omnes vero invenies univerſam motus theoriam indeterminatis et vacillantibus infinite parvorum notionibus principiisque ſuperſtructam, conſequenter evidentia illa et certitudine deſtitutam, qua diſquiſitiones mathematicae ſe tantopere ſolent commendare.

Quodſi autem partem adplicatam ſcientiarum ſtatico-mechanicarum ſeorſim attentius expendamus, in qua generalia principia aequilibrii et motus in parte pura ſtabilita ad vires in natura re ipſa exiſtentes applicantur; imperfectio ejus omnem expectationem ſuperabit, niſi nos ad difficultates fere inſuperabiles continuo reflectamus, quae in excolenda hac parte ſe perpetuo offerunt. Tot ſane tantisque tenebris veritas involuta hic deliteſcit, tam abdita, et ab oculis noſtris tam longe remota, ut, ſi vel adproximare ad illam cupias, multo pluribus nobilioribusque obſervationibus et experientiis te opus habere facillime ſentias, quam ad hunc uſque diem facere capereque licuit. Atque hinc factum eſt, quod pleraeque adplicationes

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hypothesibus physicis sint superstructae, quae jam nulli genuino, jam admodum levi fundamento innituntur, jam etiam inter se plurimum dissentiunt, et saepenumero cum veritatibus et principiis extra omne dubium positis evidenter videntur pugnare.

Haec sunt, quae me ad elaborationem hujus operis excitarunt, in quo quaecunque hactenus in universis scientiis statico-mechanicis praeclare acta sunt, ita exponere consitui, ut inde plurimum et utilitatis tyrones, primam, ut ajunt, harum scientiarum studio manum admoventes, et commodi Doctores, lucrique ipsae scientiae queant capere, unde sequens totius operis ratio et oeconomia sponte se obtulit.

1. Praemitto in hoc primo opusculorum volumine elementa calculi differentio-integralis, et geometriae sublimioris, in quorum expositione ita versor, ut cuncta ex evidentissimis analyseos finitorum notionibus et principiis elementaribus deducam, notionibusque infinite parvorum, quorum justi osores sunt recentissimi geometrae, perfectissime exclusis, omnia ad genuina principia methodi exhaustionis veterum revocem, qua de re uberius in introductione dissero.

2. His jam elementis superstruam omnes disquisitiones statico-mechanicas, quas sequentia volumina complectentur. Dabo operam, ut, inchoando a primis staticae et mechanicae principiis, quaecunque usu aliquo et utilitate se commendare visa fuerint, in diversis dissertationibus criticisque commenta-

tionibus ſolide aeque ac ſuccincte exponantur illuſtrenturque. Et ideo, ubicunque per naturam objecti licuerit, conabor ordinem notionum propoſitionumque ſcientificum, et rigorem in condendis demonſtrationibus geometricum ſervare. In ordinandis vero diſſertationibus ad id ſolum attendam, ne ulla innitatur principiis, quae nondum ſint expoſita et evicta. Non enim aedificium hic ad omnes leges architecturae exactum ſtruere, ſed materiam duntaxat conquirere mihi propoſui, qualem amplitudo et dignitas ejusmodi aedificii poſtulat: nihil ergo refert, quo ordine ſingulae ejus partes expendantur, modo materia pro ſingulis, quoad licuerit, ſolida determinetur.

3. Eapropter adplicatam ſtatico-mechanicarum ſcientiarum partem, et in ſpecie doctrinam de aequilibrio virium in machinis machinarumque motu peculiariter diſcutiam. Syſtemata hypotheſibus phyſicis ſuperſtructa, ne ſine neceſſitate ſim infinitus, breviter recenſebo; tum ſingula tali examini ſubiiciam, quod ad explorandam veritatem ſufficiat. Quamobrem non illa ſolum, quae hactenus ſunt inſtituta, experimenta conſulam, ſed alia quoque copioſa et exquiſita in ſubſidium vocabo, quorum pleraque in machinis jam exſtructis capere conſtitui.

4. In ſecundo itaque volumine, quo ad omnis generis diſquiſitiones tyrones praeparem, agam de generalibus corporum viriumque in illa agentium proprietatibus; virium aequilibrio, earumque compoſitione et reſolutione; centro aequilibrii

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in genere, et centro gravitatis ejusque investigatione in specie; motu punctorum aequabili et inaequabili; motu massarum progressivo pendente a datis viribus; lapsu corporum gravium per plana inclinata, et lineas curvas; denique de momento inertiae, motuque massarum circa axes fixos, et motu oscillatorio pendulorum. In tertio autem volumine evolvam, ob eandem caussam, principia generalia tam aequilibrii, quam motus in machinis simplicibus et compositis, ita ut hinc ad aequilibrium et motum pro determinatis viribus facillimum sit transire, felicissimoque successu principia utriusque constabilire, modo leges virium in natura existentium, quas eae sibi nullis calculis praescribi patiuntur, ex observationibus et experimentis colligere liceat.

Tuum jam erit Lector benevole judicare, utrum haec aliquid, an nihil meriti habeant. Si nihil inter ea novi, quod meum sit, inveneris; spero tamen fore, ut observes, me aliorum meditationes haud temere congessisse, sed plerasque ita excoxisse, ut meae esse videantur.

Dabam Lipsiae 17mà. 7bris. An. 1799.

INDEX CAPITUM

IN HOC VOLUMINE CONTENTORUM.

INTRODUCTIO

IN CALCULUM DIFFERENTIO-INTEGRALEM.

I.

Theorema elementare, cujus demonſtrationem in (129. §.) dedi, eſt unum e praecipuis principiis generalibus *methodi exhauſtionis* veterum geometrarum. Demonſtratio aequalitatis duorum quantorum illi principio rite ſuperſtructa tantam parit evidentiam et certitudinem, tantamque habet vim in intellectu de veritate convincendo, ut ea major deſiderari non poſſit. Qui aliter ſentiunt, non ſatis mihi videntur naturam ejusmodi demonſtrationum expendiſſe. Inter hos *Michelſenus* eſt, qui negat, methodum exhauſtionis veterum per ſe ad veritates maximo, qui poſſit deſiderari, rigore evincendas ſufficere (a): aſt non eſt, cur in confutanda ejus opinionis levitate diutius immorer; unico exemplo utar, quo clarius tyrones naturam demonſtrationum hujus generis perſpiciant. Sit nimirum demonſtranda aequalitas inter aream circuli et productum ex ejus ſemiperipheria in radium.

A) Radius circuli ſit r et π ejus peripheria, X autem area. Si p denotet perimetrum cujuscunque polygoni regularis circulo inſcripti, et P perimetrum polygoni ſimilis eidem circulo circumſcripti; erit evidenter tam $\pi > p$ et $\pi < P$, quam $X > \frac{1}{2} p r$ et $X < \frac{1}{2} P r$.

B) Sint $D = P - \pi$ et $d = \pi - p$ differentiae inter peripheriam circuli et perimetros polygonorum in (A); erit $P = D + \pi$, et $p = \pi - d$: igitur in (A) erit etiam $X > \frac{\pi r}{2} - \frac{d r}{2}$ et $X < \frac{\pi r}{2} + \frac{D r}{2}$.

(a) Beyträge zur Beförderung des Studiums der Mathematik. 3tes Stück. S. 193. 194.

C) Clarum porro eſt, rectas aequales differentiis D, d in (B) eo minores fieri, quo propius polygona in (A) accedunt ad circulum, neque difficulter demonſtratur, pro quavis cogitabili lineola recta z talia polygona in (A) eſſe poſſibilia, pro quibus quaevis rectarum D, d minor ſit lineola z, conſequenter etiam $\frac{Dr}{2} < \frac{zr}{2}$, $\frac{dr}{2} < \frac{zr}{2}$.

D) Porro nulla eſt cogitabilis tam parva ſuperficies e, pro qua et quovis dato radio r non ſit poſſibilis ejusmodi lineola recta z, quae det $\frac{1}{2}rz < e$. Cum enim abſurdum ſit de ſuperficie e cogitare, quin in hac tria puncta exiſtant, quae nec in una recta jaceant, neque in unum punctum confluant; evidenter patet, pro quavis quantumlibet parva ſuperficie e poſſibile eſſe triangulum, puta altitudinis a et baſeos b, quod minus ſit ſuperficie e, nimirum $\frac{1}{2}ab < e$. Quantumcunque porro parvae ſint lineolae b, a, cum illas cum punctis confundere non liceat; neceſſe eſt, ut pro dato quovis radio r poſſibilis ſit quarta proportionalis z ad r, b, a, quo fiat $r:b = a:z$, hinc $rz = ba$, et $\frac{1}{2}rz = \frac{1}{2}ba$; proinde etiam $\frac{1}{2}rz < e$.

E) Quare evidens eſt, ſemper eſſe poſſibilem lineolam rectam z, pro qua dimidium $\frac{1}{2}rz$ rectanguli rz minus ſit quavis utcunque parva ſuperficie e per (D): igitur erunt poſſibiles rectae D, d, pro quibus ſit tam dimidium $\frac{1}{2}Dr$ rectanguli Dr, quam dimidium $\frac{1}{2}dr$ rectanguli dr minus quavis cogitabili ſuperficie e per (C), quod eſt impoſſibile, quin ſit $X = \frac{1}{2}\pi r$ ob (B) per (129. §.).

2. Hoc modo procedes, quotiescunque principium in (129. §.) demonſtratum ad evincendam aequalitatem duorum quantorum Z, X applicare volueris. Quaere ante omnia, an non dentur duo quanta variabilia U, V, pro quibus ſemper ſit $Z > X - U$ et $Z < X + V$ vel $Z < X - V$, quoscunque valores quanta U, V obtineant: tum diſquire ſedulo, utrum haec quanta ejus naturae ſint, ut quodvis illorum quolibet quanto utcunque parvo minus poſſit fieri. Si enim haec evicta fuerint, conſtabit eo ipſo, quanta Z, X inter ſe debere aequari, ita ut abſurdum ſit differentiam aliquam inter illa ſtatuere, utcunque parva ea eſſe dicatur, determinabilis, vel indeterminabilis (129. §.), qualem *Michelſenus* in praecedente exemplo a circulo petito ſibi imaginatur. Videamus jam, quanta ſit hujus methodi vis in promovenda methodo inventionis, ſi illa ad differentias functio-

functionum rite applicetur, quae disquisitio eo plus utilitatis habere censenda est, quo evidentius inde et genuinum objectum calculi differentio-integralis, et sufficientia methodi exhaustionis veterum ad constabilienda hujus calculi principia potest colligi.

3. Plerique analystae, duce *Leibnizio* calculi differentialis inventore, universum hunc calculum circa infinite parvas differentias quantitatum variabilium et functionum versari arbitrantur, cujus systematis principia palmaria ad sequentia capita possunt revocari.

A) Si a denotet quamcunque quantitatem *finitam*, qua et major et minor sit assignabilis; et m sit numerus integer quiscunque; erit $\frac{a}{m} = x$ pars *mta* quantitatis a, eaque eo minor, quo major fuerit numerus m. Data jam utcunque parva quantitate e possibilis erit numerus integer m, pro quo valor fractionis $x = \frac{a}{m}$, consequenter etiam quaevis fractionum $\frac{a}{m^2}$, $\frac{a}{m^3}$, $\frac{a}{m^4}$, etc. fiat minor quantitate e: si ergo cogitetur numerus m indefinenter crescere, licebit hac via ad numerum infinite magnum $m = \infty$, nimirum omni dabili majorem, et quantitates *infinite parvas* $\frac{a}{\infty}$, $\frac{a}{\infty^2}$, $\frac{a}{\infty^3}$, $\frac{a}{\infty^4}$, etc., seu omni dabili minores, mente adproximare, inter quas idcirco et finitas quantitates ejusmodi intercedet relatio, ut quaelibet quantitas infinite parva $\frac{a}{\infty^n}$ tam respectu quantitatis cujuscunque finitae, quam respectu infinitae parvae inferioris ordinis $\frac{a}{\infty^{n-1}}$ perfecte debeat evanescere.

B) Quamobrem licebit etiam datam quamcunque quantitatem variabilem x incremento infinite parvo $\frac{x}{\infty}$ augere, quo quaevis ejus functio y certa quantitate mutetur, nimirum crescat vel decrescat: quantitatem hanc vocant *Differentiale functionis* y, et incrementum $\frac{x}{\infty}$ variabilis x *Differentiale variabilis* x, illudque designant signo dy, et istud signo dx. *Calculus differentialis* complectitur methodum inveniendi differentialia functionum, et *Calculus integralis* methodum investigandi functiones, quarum data sint differentialia.

C) Facillimum vero est in hoc systemate differentialia quarumvis functionum investigare. Cogitetur enim variabilis x augeri incremento

 infinite

infinite parvo $dx = \frac{x}{\infty}$, et a valore, y^1 quem functio y induat, variabili x in $x + dx$ abeunte, subtrahatur ipsa functio y; residuum $y^1 - y$ dabit differentiale dy functionis y per (B), quod simplicissime exprimes, si, peracta reductione residui $y^1 - y$, summis formae $\alpha + \beta dx + \gamma dx^2 + \delta dx^3 +$ etc. quantitates finitas α, et summis formae $\beta dx + \gamma dx^2 + \delta dx^3 +$ etc. quantitates infinite parvas βdx substituas, ob (B et A).

E. gr. Si petas differentiale dy functionis $y = ax^3$; erit

$$y^1 = a(x + dx)^3 = ax^3 + 3ax^2dx + 3axdx^2 + adx^3:$$

$$\text{igitur } y^1 - y = 3ax^2dx + 3axdx^2 + adx^3,$$

$$\text{seu } dy = 3ax^2dx$$

Pro $y = \frac{x}{1 - x^2}$ autem erit

$$\frac{x + dx}{1 - (x + dx)^2} - \frac{x}{1 - x^2} = y^1 - y = dy:$$

$$\text{igitur } dy = \frac{(1 + x^2)dx + xdx^2}{(1 - x^2)^2 - 2x(1 - x^2)dx - (1 - x^2)dx^2}$$

$$\text{seu } dy = \frac{(1 + x^2)dx}{(1 - x^2)^2}.$$

4. Nihil hoc calculo simplicius poterat excogitari, seu inventionem differentialium datarum functionum, seu illarum, quae incognitis functionibus competant, investigationem consideres: verum fundamenta ejus acerrimo inter ipsos illorum patronos bello occasionem praebuere, ad quod generatim inter analystas excitandum fovendumque notiones infinite parvorum aptissimae esse videntur. Sunt, qui duce *Eulero* omnes quantitates infinite parvas nihilo aequari judicant, ita tamen, ut eo non obstante diversae rationes geometricae inter ipsas possint intercedere. Aliis contra absurdus esse videtur nihilorum calculus, absurdumque, ut una duarum quantitatum infinite parvarum, inter quas, utpote aequales nihilo, nulla intercedat magnitudinis differentia, *ntupla* possit esse alterius. Quamobrem propugnant ii, omnem quantitatem infinite parvam esse quantum sui generis, omni quidem dabili minus, non tamen aequale nihilo: nec metuunt, ne ob neglectum infinite parvorum respectu finitorum infiniteque parvorum inferioris ordinis (3) calculus erroneus reddatur; error enim, inquiunt, qui hic committi videtur, est infinite parvus, consequenter

quenter omni dabili minor, et ideo nullus. Ast, ne nimiopere a fcopo hujus introductionis abducamur, affumamus explorandam vim methodi exhauftionis veterum in ufu differentiarum finitarum, quo evidentius pateat, utrum et qua ratione calculi differentio-integralis principia ad genuina methodi exhauftionis veterum principia poffint revocari, quod unico exemplo illuftraffe fufficiet.

5. *Pro quibuslibet duobus quantis* P, Q *dabitur numerus integer* m, *quo fumma* $P\frac{1}{m}+Q\frac{1}{m^2}$ *fiat minor quavis utcunque parva quantitate* e.

Semper enim erit tertium quantum Z poffibile, quod majus fit quolibet quanto P, Q feorfim fumto: erunt ergo quanta Z, P, Q fic comparata, ut pro illis fit $Z\frac{1}{m}+Z\frac{1}{m^2}>P\frac{1}{m}+Q\frac{1}{m^2}$.

Cum vero fit $Z\frac{1}{m}>Z\frac{1}{m^2}$, adeoque $Z\frac{1}{m}+Z\frac{1}{m}>Z\frac{1}{m}+Z\frac{1}{m^2}$; erit eo ipfo etiam $2Z\frac{1}{m}>P\frac{1}{m}+Q\frac{1}{m^2}$.

Quantumcunque porro magnum fit quantum $2Z$, et utcunque parvum e, poffibilis erit numerus integer m, pro quo fit $2Z\frac{1}{m}<e$: pro eodem numero erit ergo a fortiori $e>P\frac{1}{m}+Q\frac{1}{m^2}$.

6. *Data aequatione* $y^2=Ax\pm Bx^2$ *ad curvam* BPC (18. Fig.) *inter ejus coordinatas orthogonas* $y=Pa$, $x=Ba$, *cujus revolutione circa axem* BE *generetur corpus rotundum* CBD; *invenire foliditatem fegmenti* BPp *plano* PmpnP *ad axem* BE *perpendiculari abfciffi.* Fig. 18.

Solutio. I. Soliditas quaefita fit S = BPnpmP, quae variabili $x=Ba$ crefcente quacunque differentia $\Delta x=ab$ crefcet differentia $\Delta S=$ Pmpncserc: erit ΔS major cylindro bafeos Pmpn et altitudinis $ab=\Delta x$, et fimul minor cylindro bafeos resc et altitudinis $ab=\Delta x$: cum igitur, pro ratione radii ad femiperipheriam $=1:\pi$, fit $\pi.Pa^2.\Delta x$ foliditas cylindri prioris, et $\pi.rb^2.\Delta x$ foliditas pofterioris; debebit effe femper

$$\Delta S>\pi.Pa^2.\Delta x;\ \text{et}\ \Delta S<\pi.rb^2.\Delta x.$$

II.

II. Sed per hypothesim debent ordinatarum Pa, rb abscissis $x=Ba$, $x+\Delta x=Bb$ respondentium quadrata esse $Pa^2=Ax\pm Bx^2$, $rb^2=A(x+\Delta x)\pm B(x+\Delta x)^2$; igitur erit semper etiam

$$\Delta S>\pi(Ax\pm Bx^2)\Delta x;$$
$$\text{et } \Delta S<\pi(A(x+\Delta x)\pm B(x+\Delta x)^2)\Delta x.$$

III. Si porro haec omni casu debent subsistere, quemcunque valorem habeat differentia $\Delta x=ab$; necesse est, ut eadem locum habeant, si, variabili $x=Ba$ in numero m partes aequales mente divisa, differentia $\Delta x=ab$ uni m*tae* parti ejusdem variabilis aequetur, adeoque ponatur $\Delta x=\frac{x}{m}$.

IV. Quodsi autem Ba in numero m, et Bb in $m+1$ partes aequales mente dividatur, quarum quaevis sit $=\Delta x$; et per singula puncta divisionis cogitentur duci sectiones ad axem perpendiculares, ut $PmpnP$, $rescr$ per puncta a, b; dividetur eo ipso $S=BPnpmP$ in numero m partes, et $Brcser=S+\Delta S$ in partes numero $m+1$, quae sic debebunt esse comparatae, ut, si in ΔS terminos seriei $\Delta x.0, \Delta x.1, \Delta x.2, \text{---} \Delta x(m-1)$ loco x successive substituas, numero m partes segmenti $S=BPnpmP$ ordine sis obtenturus.

V. Hinc ergo (IV) et ex (II) evidenter elucet, soliditatem S pro quovis possibili numero integro m in (III) majorem esse summa omnium valorum, quos functio

$$\pi(Ax+Bx^2)\Delta x$$

successive induet, si in illa termini seriei $\Delta x.0, \Delta x.1, \Delta x.2, \text{----} \Delta x(m-1)$ loco x successive substituantur; et simul minorem summa omnium valorum, quos in eadem hypothesi functio

$$\pi(A(x+\Delta x)\pm B(x+\Delta x)^2)\Delta x$$

debebit obtinere. Quare habebimus:

$$S>\pi\begin{pmatrix} A(1+2+3+\text{---}+(m-1))\Delta x^2 \\ \pm B(1^2+2^2+3^2+\text{---}+(m-1)^2)\Delta x^3 \end{pmatrix}$$

$$S<\pi\begin{pmatrix} A(1+2+3+\text{-----}+m)\Delta x^2 \\ \pm B(1^2+2^2+3^2+\text{-----}+m^2)\Delta x^3 \end{pmatrix}$$

Et

Et sumtis serierum summis fiet

$$S > \pi\left(\tfrac{1}{2}A(m^2 - m)\Delta x^2 \pm \tfrac{1}{6}B(2m^3 - 3m^2 + m)\Delta x^3\right);$$
$$S < \pi\left(\tfrac{1}{2}A(m^2 + m)\Delta x^2 \pm \tfrac{1}{6}B(2m^3 + 3m^2 + m)\Delta x^3\right).$$

Cumque sit $\Delta x = \frac{x}{m}$; erit etiam

$$S > \tfrac{1}{2}\pi A x^2 \pm \tfrac{1}{3}\pi B x^3 - U;$$
$$\text{et } S < \tfrac{1}{2}\pi A x^2 \pm \tfrac{1}{3}\pi B x^3 + V;$$
$$\text{pro } U = \tfrac{1}{2}\pi x^2 (A \pm B x)\frac{1}{m} \mp \tfrac{1}{6}\pi B x^3 \frac{1}{m^2};$$
$$\text{et } V = \tfrac{1}{2}\pi x^2 (A \pm B x)\frac{1}{m} \pm \tfrac{1}{6}\pi B x^3 \frac{1}{m^2}.$$

VI. Hinc, quia quodvis quantorum U, V in (V) potest fieri minus quolibet utcunque parvo quanto e per (5), necessario sequitur esse $S = \frac{1}{2}\pi A x^2 \pm \frac{1}{3}\pi B x^3$ per (2).

7. Si utramque expressionem in (6. n. II.) per Δx dividamus, obtinebimus pro quotiente $\frac{\Delta S}{\Delta x}$ sequentes conditiones, posita differentia $\Delta x = \frac{x}{m}$.

$$\frac{\Delta S}{\Delta x} > \pi(A x \pm B x^2);$$
$$\frac{\Delta S}{\Delta x} < \pi(A x \pm B x^2) + (A \pm 2 B x) x \frac{1}{m} \pm B x^2 \frac{1}{m^2}.$$

Ob primam conditionem debet necessario extare certa differentia

$$\frac{\Delta S}{\Delta x} - \pi(A x \pm B x^2) = D, \text{ pro qua sit}$$
$$\frac{\Delta S}{\Delta x} = \pi(A x \pm B x^2) + D:$$

igitur ob secundam conditionem

$$D < (A \pm 2 B x)\, x \frac{1}{m} \pm B\, x^2 \frac{1}{m^2}.$$

Quamobrem, cum id semper debeat subsistere, utcunque parva sit pars *mta* $\Delta x = \frac{x}{m}$ variabilis x, quamtumlibet idcirco magnus sit nu-

merus integer m; cumque semper sit numerus m possibilis, pro quo $(A \pm 2Bx)x\frac{1}{m} \pm Bx^2\frac{1}{m^2}$ valorem minorem quovis utcunque parvo valore e obtineat (5): evidens est, differentiam D inter $\frac{\Delta S}{\Delta x}$ et $\pi(Ax \pm Bx^2)$ minorem quavis utcunque parva quantitate e posse fieri, consequenter functionem $\pi(Ax \pm Bx^2)$ esse limitem omnium mutationum, quas exponens $\frac{\Delta S}{\Delta x}$ rationis $\Delta S : \Delta x$ successive potest subire, si $\Delta x = \frac{x}{m}$ indesinenter decrescat, numero m perpetuo crescente (68. §. Elem.).

8. Pari ratione, si opus esset, liceret usum principiorum methodi exhaustionis veterum ad omnes casus extendere, ad quos fundamentalia calculi differentio-integralis praecepta solent adplicari, quin sit necessarium ad notiones infinite parvorum refugere, quorum ne vestigium in ejusmodi disquisitionibus reperies. Ex hoc autem unico exemplo evidenter colligimus, possibile esse, ut principia calculi differentio-integralis eadem certitudine et evidentia geometrica, quam in Archimedeis disquisitionibus admiramur, constabiliantur, si objectum hujus calculi ita determinetur, ut ille perpetuo circa solos limites versari debeat, ad quos exponentes rationum $\Delta y : \Delta x$ inter differentias Δy functionum y et differentias Δx variabilium absolutarum x (4. §. Elem.) intercedentium indesinenter adpropinquent, differentiis Δx continuo decrescentibus; tum ad evolutionem eorundem limitum et proprietatum, quibus ii sint praediti, methodus exhaustionis veterum rite applicetur. Limites ejusmodi genuinum esse, idque unicum rationi consentaneum, objectum calculi differentialis, communis est celeberrimorum Analystarum sententia. Alii illam diserte profitentur, principiaque calculi differentialis ex ipsis talium limitum notionibus derivant: alii vero eam tacite amplectuntur, inter quos perspicacissimus *La Grangius* eminet. Negat enim *La Grangius* in sua theoria functionum analyticarum (a), principia calculi differentialis certitudine evidentiaque geometrica constabiliri, si illa ipsis limitum notionibus superstruantur; quocirca alia is methodo, independente ab his notionibus, functiones

(a) *Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toute considération d'infiniment petits ou d'évanouissans, de limites ou de fluxions, et réduits à l'analyse algebrique de quantités finies.*

a diversorum ordinum ex quavis functione primitiva evolvere docet, quae nihil sunt aliud, quam limites mutationum in exponentes $\frac{\Delta y}{\Delta x}$, $\frac{\overset{2}{\Delta} y}{\Delta x^2}$, $\frac{\overset{3}{\Delta} y}{\Delta x^3}$, etc. ob differentiam Δx continuo decrescentem redundantium: eapropter ipse *La Grangius* suum *calculum functionum* eundem cum *calculo differentiali* esse pronunciat, quod *Ill. La Croix* in suis institutionibus calculi differentio-integralis (a) insigniter illustravit, *La Grangianum* calculum ad evolutionem principiorum calculi differentialis adplicans.

9. Verum veniam dabit Vir celeberrimus, si ei in re tanti momenti contradicam, notionesque genuinas limitum, de quibus hic sermo est, tales esse judicem, ut ex illis principia calculi differentialis omni rigore certitudineque et evidentia, cujus disquisitiones analyticae capaces sunt, possint derivari, ita ut calculus hic, iisdem notionibus superstructus, totus elementaribus principiis analyseos finitorum innitatur. Quicunque enim hactenus id praestare adnisi sunt, in eo mihi peccasse videntur, quod a definitionibus rationum differentialium immediate ad illarum investigationem transiverint, et saepenumero in hac investigatione notionibus sint usi, quae cum rigore, quo principia calculi differentialis erant constabilienda, nullo pacto possunt conciliari. *Clarissimum L'Huilierum* inter recentiores excipio, qui ante omnia generales limitum pro quantis rationibusque variabilibus proprietates evolvere est conatus, quo fundamenta solida constabiliret, quibus investigatio rationum differentialium posset inniti (b). Ast utrum haec aliquid et quantum ponderis habeant, ex sequenti commentatiuncula elucebit, quae et ipsam methodum, qua in constabiliendis principiis calculi differentio-integralis utor, et ejus vim in evincendis veritatibus illustrabit.

c 2 10. Bina

(a) *Traité du Calcul differential et intégral.*

(b) *Exposition élementaire des principes de calculs superieurs, qui a remporté le prix proposé par L'Académie Royale de Sciences et Belles-Lettres pour L'Année 1786. à Berlin.*

Principiorum calculi differentialis et integralis expositio elementaris ad normam dissertationis ab academia Scient. Reg. Prussica anno 1786. praemii honore decoratae elaborata. Tubingae.

10. Bina extant principia maximi momenti in universa functionum theoria, de quorum solida demonstratione parum hactenus erant Analystae solliciti: primum est, quod omnis functio variabilis z certae seriei formae generalis $Az^a + Bz^b + Cz^c +$ etc. aequetur: alterum vero, quod differentia Δy cujuslibet functionis y variabilis z alicui seriei formae $\alpha \Delta z + \beta \Delta z^2 + \gamma \Delta z^3 +$ etc. debeat esse aequalis. Neutrum principium est per se evidens: utrumque ego ideo in mea analysi (a) demonstrare adnisus sum, quo institutiones calculi differentio-integralis firmis fundamentis possent superstrui. Neglexit hoc *L'Huilierus*, quamvis is utroque principio ad evincendum theorema Taylorianum usus sit, quod eo minus possum adprobare, quo amplior est apud illum usus hujus theorematis (b). *La Grangius* contra, ne a nondum evicto principio suam functionum theoriam ordiatur, demonstrare conatur, pro quavis functione fx variabilis x, si $f(x+i)$ denotet valorem, quem illa obtineat, variabili x in $x+i$ abeunte, esse $f(x+i) = fx + pi + qi^2 + ri^3 +$ etc., ita ut in hac serie solae potentiae exponentium integrorum et positivorum incrementi i possint contineri (c): ast multa mihi in hac demonstratione displicent.

1°. Tota demonstratio aequationum theoriae innititur, quae ob exponentes potentiarum $i, i^2, i^3,$ etc. in infinitum crescentes nequit hic eam evidentiam et certitudinem parere, qua fundamentale totius calculi functionum principium erat demonstrandum.

2°. Porro est tota demonstratio tantum ad impossibilitatem potentiarum fractarum $i^{\frac{m}{n}}$ evincendam concinnata, quin inde ulla ratione possit elucere, utrum series exprimens functionem $f(x+i)$ terminum aliquem pi primam potentiam incrementi i complectentem necessario debeat habere, cum tamen, nisi ejusmodi terminus adsit, universus calculus vacillet.

3°. Ponamus demum, *La Grangium* solidissime evicisse non posse fieri, ut $f(x+i)$ aequetur alicui seriei formae $fx + pi^a + qi^b + ri^c +$ etc., in

(a) Unterricht in der mathematischen Analysis. 2ter Band, aufs Jahr 1791; und Beylage aufs Jahr 1798.

(b) *Principiorum Calc. Diff. et Integr. Expositio etc.* 32. 33. §.

(c) *Théorie des fonctions analytiques etc.* 10. §.

in qua exponentes fracti occurrant: estne ideo evictum, functionem $f(x+i)$ aequari seriei $fx+pi+qi^2+ri^3+$etc.? minime: nihil enim aliud inde sequitur, quam quod nullus exponentium a, b, c, etc. possit esse fractus, *si functio* $f(x+i)$ *certae seriei formae* $fx+pi^a+qi^c+ri^c+$etc. *aequetur;* utrum autem re ipsa extet ejusmodi series, nullibi demonstratum invenio.

11. Espropter, ut tyrones a primis vulgaris algebrae elementis ad adaequatam distinctissimamque cognitionem principiorum calculi differentio-integralis via brevissima ducerem, exposui in primo elementorum hujus calculi capite palmares functionum proprietates, explicationemque functionum algebraicarum, logarithmicarum, exponentialium et trigonometricarum per series methodo simplicissima docui; tum, qua ratione principiorum, de quibus in (10) erat sermo, veritas potest evinci, in (63. 66. §.) ostendi. Quod in specie ad theorema binomiale attinet, habes illud pro quolibet exponente rationali et irrationali m in (51. §.) methodo inventionis demonstratum, ita ut adversus hujus simplicissimae demonstrationis rigorem nihil possit obiici. Cardo enim totius demonstrationis in (n. 3.) consistit, ubi generatim supposui, arithmeticam universalem ita esse excultam, ut inde facile eluceat, cur pro omni exponente m, et quibuslibet quantis u, v, debeat esse $\frac{u^m}{v^m}=\left(\frac{u}{v}\right)^m$: qui de hujus principii veritate dubitat, neque concipere potest, quo modo id demonstrare liceat; praecipua quaeque principia arithmeticae universalis, ut e. gr. $a^m b^m=(ab)^m$; $a^m a^n=a^{m+n}$, $\text{Log}\, r^m=m \log r$; $\log\frac{r}{s}=\log r-\log s$; et similia in dubium vocet oportet, quae utique nequeunt subsistere pro omnibus exponentibus, quin sit e. gr. $\log\frac{u^m}{v^m}=\log u^m-\log v^m=m\log u-m\log v=m\log\frac{u}{v}=l\left(\frac{u}{v}\right)^m$, adeoque necessario $\frac{u^m}{v^m}=\left(\frac{u}{v}\right)^m$.

12. Hinc jam ad ipsam expositionem principiorum calculi differentialis in secundo capite progredior. Ante omnia expono adoptatas et methodo exhaustionis veterum consentaneas definitiones *limitum* pro quantis et rationibus variabilibus (68. 69. §.), ex quibus evidentissime principia generalia sequuntur, quae usque ad (78. §.) evolvo. Tum, praemissa definitione rationis differentialis (79. §.), assumtoque pro ejus exponente signo simplicissimo, totus in eo occupor, ut illas rationum differentialium proprietates generales paucis explicem et evincam, quibus universus

 diffe-

differentio-integralis calculus innititur (80 - - - 102. §.): ex his demum proprietatibus praecepta universi calculi differentialis per se indidem evidentissime fluentia elicio (103 - - - 159. §.). Peculiarem vero operam adhibui, ut universalem eamque planissimam methodum quaerendi exponentes rationum differentialium functionibus incognitis debitarum docerem, illamque ad ejusmodi principia revocarem, de quorum certitudine evidentiaque dubitare sit impossibile (129 - - - 132. §.) Quantam haec principia vim in evincendis veritatibus habeant, si illa ad geometricas disquisitiones rite applicentur, ostendi in postremis duobus capitibus: possunt autem illa eadem facilitate perspicuitateque ad generalia totius mechanicae fundamenta constabilienda adplicari, quin necessarium sit, tempora, spatia, celeritates, et similia in statu evanescentiae considerare, illaque ut infinite parva vel evanescentia inter se comparare.

13. Hoc modo investigatio rationum differentialium ad duo generalissima principia analytica revocatur (82. 131. §.). Vel enim functio y relata ad varjabilem absolutam x (4. §.) datur, vel penitus ignoratur. Casu primo ponatur in illa ubique $x+\Delta x$ loco x, quo ipso obtinebitur differentia $\Delta y = y^{I} - y$. Haec jam differentia ita transformetur, ut perspicue constet, ipsam certae seriei formae $P\Delta x + A\Delta x^2 + B\Delta x^3 + C\Delta x^4 +$ etc. aequari, in qua terminus primus $P\Delta x$ perfecte sit determinatus, seu coefficientes A, B, C, etc. reliquorum terminorum noscantur seu non, modo ii a differentia Δx sint independentes. Quodsi enim reipsa sit $\Delta y = P\Delta x + A\Delta x^2 + B\Delta x^3 +$ etc.; erit eo ipso $\partial y = P = P\partial x$ exponens rationis differentialis functionis y (82. 81. §.).

14. Si autem ignoretur functio y variabilis absolutae x; ignorabitur quoque expressio illius differentiae Δy. Hoc casu investigetur, annon existent binae functiones U, V differentiae Δx, inter quas incognita differentia Δy sic jaceat, ut pro quovis utcunque parvo valore differentiae Δx sit

$$\Delta y > U \text{ et simul } \Delta y < V.$$

Tum determinentur expressiones earundem functionum U, V per x, Δx, et quantitates constantes, a quibus illae dependeant: expendantur eaedem expressiones diligentissime, ut certo constet, an non detur determinata aliqua quantitas X, constans vel aequalis certae functioni variabilis x,

modo

modo illa fit independens a differentia Δx; et an non fint praeterea aliae quantitates $e, f, g, ---- E, F, G,$ etc. poffibiles, quae a Δx non dependeant (licet eae ignorentur) ficque fint comparatae, ut pro illis nafci deberent feries $X\Delta x + e\Delta x^2 + f\Delta x^3 + g\Delta x^4 +$ etc., $X\Delta x + E\Delta x^2 + F\Delta x^3 + G\Delta x^4 +$ etc. functionibus U, V aequales, quibus in praecedentibus expreffionibus pro Δy fubftitutis fieret

$$\Delta y > X\Delta x + e\Delta x^2 + f\Delta x^3 + g\Delta x^4 + \text{etc.}$$

et fimul

$$\Delta y < X\Delta x + E\Delta x^2 + F\Delta x^3 + G\Delta x^4 + \text{etc.}$$

Utprimum enim hujusmodi expreffiones fuerint detectae, certoque conftiterit, utramque fimul pro quovis quantumlibet parvo valore differentiae Δx debere fubfiftere; certum etiam erit, X effe exponentem rationis differentialis incognitae functionis y, nimirum $\partial y = X = X \partial x$ (131. §.).

Sic e. gr. fuperius (6), licet ignoretur functio $S = BPnpmP$ variabilis $x = Ba$, certo tamen conftat, illius differentiam ΔS ejusmodi effe, ut pro quovis valore differentiae Δx fit ΔS major cylindro bafeos $Pmpn$ et altitudinis Δx, et fimul minor cylindro bafeos resc et altitudinis Δx, unde ibidem obtinuimus

$$\Delta S > \pi(Ax \pm Bx^2)\Delta x;$$

et fimul

$$\Delta S < \pi(A(x+\Delta x) \pm B(x+\Delta x)^2)\Delta x.$$

Quamobrem erit etiam

$$\Delta S > \pi(Ax \pm Bx^2)\Delta x;$$

et fimul

$$\Delta S < \pi(Ax \pm Bx^2)\Delta x \pm 2Bx\Delta x^2 \pm B\Delta x^3.$$

Atque ex his expreffionibus evidenter jam per (131. §.) fequitur, exponentem rationis differentialis incognitae functionis S neceffario effe

$$\partial S = \pi(Ax \pm Bx^2) = \pi(Ax \pm Bx^2)\partial x,$$

quod ipfum, ob (79. 70. §.), jam fuperius (7) invenimus.

15. Quod ad principia in (95 - - - 102. §.) demonftrata in fpecie attinet, ea nec per fe funt evidentia, ut demonftratione non habeant opus; neque

neque talia, ut ignorari possint, quin disquisitiones differentio-integrales eo ipso dubiae reddantur.

Sicut e. gr. investigatio logarithmorum datis numeris respondentium, et numerorum datis logarithmis debitorum, in determinato quovis casu dubia deberet reddi, nisi constaret, uni eidemque numero unicum logarithmum, et vicissim cuivis logarithmo unum determinatum numerum, inaequalibus vero logarithmis inaequales numeros, et numeris inaequalibus inaequales logarithmos respondere: ita etiam incerta erit quovis determinato casu investigatio exponentium rationum differentialium pro certis functionibus, et harum pro datis exponentibus, nisi similia principia pro functionibus et illarum rationibus differentialibus evincantur, ut in (95 -- 98. §.).

In (14) invenimus $\partial S = \pi(Ax \pm Bx^2)$ pro exponente rationis differentialis functionis S aequalis soliditati corporis BPnpmP: si jam hunc exponentem ad detegendas certas proprietates ejusdem corporis adplicem; certitudo investigationis in eo profecto fundabitur, quod evidenter constet, illum exponentem ei duntaxat functioni posse respondere, quae soliditati corporis BPnpmP aequatur. Et si inde ope calculi integralis ipsam soliditatem eliciam, ponamque illam esse $S = \pi(\frac{1}{2}Ax^2 \pm \frac{1}{3}Bx^3) + C$; erit haec determinatio ideo certa et evidens, quia certo constat, dato exponenti $\partial S = \pi(Ax \pm Bx^2)$ nullam aliam functionem posse respondere (97. 98. §.).

Deinde si quacunque methodo exponentem $\partial y = \alpha \partial z$ rationis differentialis functionis y datae per variabilem z determines; nisi principia in (99 - - 102. §.) evicta sint, haud licebit $\partial z = \frac{\partial y}{\alpha}$ inde derivare, et $\frac{\partial y}{\alpha}$ pro exponente rationis differentialis quantitatis z spectatae instar unius functionis variabilis y habere, quod tamen in quamplurimis disquisitionibus theoretico-practicis solet fieri.

Ea propter expositionem horum principiorum (95 --- 102. §.) non utilem duntaxat, sed etiam necessariam esse arbitror, incauteque illam hactenus a plerisque Analystis neglectam esse censeo: quodcunque systema amplectaris, similia principia necessario erunt tibi evincenda, nisi universum systema vacillantibus fundamentis velis superstruere.

16. Quamobrem id me in primis duobus capitibus perspicue ostendisse judico, principia universi calculi differentio-integralis ex ipsis limitum notionibus, independenter ab omni infinite parvorum evanescentiumque notione,

notione, sic posse derivari, ut illa firmissimis fundamentis analyseos finitorum innitantur, idque methodo aeque simplici ac universali posse praestari. Seu enim generalem theoriam ibidem expositam (68 - - 102. §.), seu illius adplicationem ad investigationem exponentium rationum differentialium datis quibuscunque functionibus, algebraicis, logarithmicis, exponentialibus, et trigonometricis, atque ex his quoque modo compositis debitarum (103 - - - 126. §.) attente expendas; seu demum methodum, qua exponentes differentiales functionibus incognitis respondentes investigo (14), ad examen voces: nullibi ne vestigium aut incrementorum infinite parvorum vel evanescentium, aut rationum inter illa intercedentium observabis, nisi notionibus adaequate determinatis, rationique et methodo exhaustionis veterum geometrarum consentaneis, quibus perpetuo utor, alias peregrinas substituas, de quibus eo minus mihi vel somniare licuit, quo minus ese cum ratiociniis, quibus singula principia evincere conatus sum, possunt conciliari.

17. Superius jam animadverti, *functiones derivatas*, quae objectum *La Grangiani* calculi functionum constituunt, nihil aliud esse, quam exponentes rationum, quas ego cum aliis Analystis in his elementis *differentiales* voco: nisi ergo de nomine litem velis movere, facile admittes, circa hos ipsos exponentes eundem calculum versari, eo duntaxat discrimine, quod *La Grangius* neque naturam illorum ex limitum notionibus determinet, neque regulas, per quas ii pro datis quibuscunque functionibus primitivis inveniri possunt, ex iisdem notionibus derivet. Verum, licet ego plene sim convictus, id tam methodo *La Grangiana*, quam alia quoque multo simpliciori, cujus specimen alia occasione dedi ([a]), posse praestari, modo et illa et haec uberius excolatur; non video, cur id sit necessarium, eo minus sane, cum certo constet, utrobique ad limitum principia esse refugiendum, utprimum calculus ad functiones incognitas adplicatur. *La Grangius* enim conatur demonstrare, valorem $f(x+i)$, quem

(a) Archiv der reinen und angewandten Mathematik; herausgegeben von *C. F. Hindenburg*. Achtes Heft, aufs Jahr 1798.

quem quaecunque functio fx variabilis x debet induere, si ipsa variabilis x capiat incrementum i, ejusmodi seriei $fx + ip + i^2q + i^3r +$ etc. aequari, in qua incrementum i sic poterit omni casu determinari, ut quivis terminus major evadat quam summa omnium sequentium terminorum: atque istud principium vocat is *theorema fundamentale* totius suae theoriae functionum, quod, ex ejus mente, tacite tam in *Leibniziano* calculo differentiali, quam in *Neutoniano* calculo fluxionum praesupponitur (b). Utramque certum est, et ego ipse ejusmodi theorema in (75. 77. §.) ex genuinis principiis (independenter ab omni curvarum theoria, cujus principia in disquisitionibus hujus generis aeque ac principia mechanicae peregrina esse censeo) evinco, tum bina inde principia generalia elicio. (82. 131. §.), eorumque uni investigationem exponentium rationum differentialium pro datis quibuscunque functionibus, alteri vero inventionem eorundem pro functionibus incognitis superstruo, ut superius (13) (14) exposui: ast an non ex his evidenter elucet, *La Grangium* investigationem suarum functionum *derivatarum*, tum praecipue, dum functiones *primitivae* ignorantur (quod plerumque fit in adplicationibus ad geometriam et mechanicam), genuinis principiis doctrinae limitum superstruere? an non ergo consultius est totum systema a limitum notionibus ordiri, iisdemque universum calculum superstruere?

18. Sed ne mihi in hac principiorum calculi differentialis expositione aliquid adtribuere videar, quod aliis acceptum debeo referre; dicam paucis, quod de hoc meo qualicunque labore sentio. Superius jam animadverti, me in determinando genuino objecto calculi differentialis aliorum sententiam amplecti: nihil ergo novi propono adferens, universum calculum differentialem circa investigationem limitum rationum inter simultanea incrementa functionum variabiliumque absolutarum, ad quas illae relatae cogitantur, intercedentium versari: id enim *Neutonus* jam docuit, docentque scriptores recentissimi, *Eulerus* (c), *Kästnerus* (d), *Karstenius*

(b) *Théorie des Fonctions Analytiques.* 14 §.

(c) *Institutiones Calculi differentialis.*

(d) Anfangsgründe der Analysis des Unendlichen.

[illegible] (e), et alii Viri celeberrimi. Cum autem ſola genuini objecti determinatio ad conſtabilienda principia calculi differentialis minime ſufficiat; nondum inde ſequitur, eosdem Viros celeberrimos haec quoque principia ita conſtabiliviſſe, ut ea certis et evidentibus fundamentis, notionibusque, nusquam peregrinis, ſed genuinis duntaxat et objecti naturae conſentaneis inniti debeant vel poſſint cenſeri. „Ego quidem contrarium obſervavi, et praecipue in adplicatione eorundem principiorum ad geometriam et mechanicam; quamobrem non tantum utile, ſed et neceſſarium eſſe arbitrabar, rigidiori examini omnia ſubiicere; doctrinam de limitibus quantorum rationumque variabilium magis excolere; generaliaque illius principia analytica, quorum veſtigia in *Käſtneri* et *Karſtenii* inſignibus operibus continuo ſe offerebant, ita conſtabilire, et methodo quoad fieri poteſt plana in unum ſyſtema redigere, ut univerſus calculus differentio-integralis nomen *calculi exactiſſimi* poſſit mereri, qui non tantum facili ampliſſimoque uſu, ſed etiam geometrica ſuorum principiorum certitudine evidentiaque ſic ſe commendet, ut non ſit cur notiones, quibus is innititur, contemni et derelinqui debeant.

19. De reliquo finis principalis, cujus gratia haec elementa conſcripſi, erat, ut tyrones ad disquiſitiones ſtatico-mechanicas praepararem, quas ſine ſublimiori analyſi nec excolere licet, neque excultas intelligere. Paucis multa exponere conatus ſum, quin ideo obſcurus fierem: omnia tamen, quae notari merebantur, explicare non poteram, ne arctos limites, intra quos me continere debebam, transgrederer. Cum autem maximum in diſquiſitionibus omnis generis pondus habeant elementa calculi differentialis, et calculi integralis reſtricti ad functiones unius variabilis; illa et haec peculiari induſtria evolvere, et ſic enodare ſtudui, ut adplicatio ipſorum facilis eſſet et commodiſſima: quapropter integralia maxime memorabilium exponentium differentialium non tantum ad generalia praecepta revocavi, ſed ſingula plene determinavi, quo in adplicatione ad caſus ſpeciales ſola ſubſtitutione eſſet

d 2 opus.

(e) Anfangsgründe der mathematiſchen Analyſis.

opus. Reliqua, ut e. gr. adplicationem calculi differentialis ad theoriam aequationum; integrationem exponentium differentialium ad plures variabiles relatorum, etc. breviſſime attigi, et geometriae ſublimioris primas tantum notiones illuſtravi. Non diffiteor itaque plurima in his elementis deſiderari, ſeu ſola principiorum Analyſeos et Geometriae ſublimioris cognitio, ſeu illorum uſus in diſquiſitionibus ſtatico-mechanicis expendatur: ſpero tamen fore, ut quicunque haec, quae expoſui, meditate perlegerint, ſibique familiaria reddiderint, facile prolixiora aliorum Scriptorum opera intelligant, in quibus hujuscemodi doctrinae ex inſtituto pertractantur: in illorum autem gratiam, qui opuſculis meis ſtatico-mechanicis uti voluerint, de iis, quae hic omittere debebam, differam, ubi iis opus habuero.

ELEMEN-

ELEMENTA
THEORIAE FUNCTIONUM,
CALCULIQUE DIFFERENTIALIS.

CAPUT I.

DE
DISCRIMINE ET DIFFERENTIIS FUNCTIONUM, EARUMQUE EXPLICATIONE PER SERIES.

1. Definitio.

Quantitates, circa quas universa analysis mathematica versatur, dividuntur in *constantes* et *variabiles:* illae eosdem perpetuo valores retinere censentur; hae vero indeterminati sunt valoris, possuntque quosvis valores recipere, quin etiam nihilo aequari: priores plerumque initialibus, posteriores autem finalibus alphabeti literis designantur.

2. Definitio.

Quaelibet expressio analytica alicujus quanti per quantitates constantes et variabiles, quomodocunque inter se connexas, vocatur *Functio* earundem variabilium: *Formam* functionis determinat ipse nexus inter quantitates constantes et variabiles, quas functio complectitur, intercedens.

3. Corollarium 1.

Omnis functio innumeros valores potest recipere, pendentes a totidem possibilibus valoribus quantitatum variabilium, quas illa continet (1. 2. §.): quaevis functio est ergo quantitas variabilis (1. §.).

4. Corollarium 2.

Quamobrem, dum certae quantitates variabiles aliarum variabilium functiones fuerint (3. §.), debebunt aliquae in se esse variabiles, independentes a novis variabilibus: quantitates hujuscemodi intelligemus deinceps nomine *variabilium absolutarum*, ad quas, unam aut plures, quaelibet functio referri censebitur.

Scho-

Scholion.

Fig. 1. Sic, si $y = PQ$ denotet perpendiculum ex indeterminato puncto P peripheriae circuli radio $r = AC$ descripti ad diametrum AB demissum, $x = CQ$ vero partem diametri inter illud perpendiculum centrumque circuli C interceptam designet; erunt y, x duae rectae ita inter se connexae, ut, alterutra illarum crescente vel decrescente, altera eo ipso mutari debeat. Si jam illae mutationes, quae ex omnibus mutationibus possibilibus distantiae x in perpendiculum y possunt redundare, in considerationem vocentur; spectabitur x instar unius variabilis absolutae, quae, independenter ab omni alia variabili, quosvis possibiles valores possit recipere; y autem erit ejus functio $= \sqrt{(r^2 - x^2)}$: si vero illarum mutationum ratio habeatur, quas distantia x debet subire, si perpendiculum y successive mutari cogitetur; referetur x ad y ut functio $= \sqrt{(r^2 - y^2)}$ ad variabilem absolutam.

5. Definitio.

Operationes algebraicas vocant ordinarias operationes arithmeticas, ut sunt, Additio, Subtractio, Multiplicatio, Divisio, Elevatio ad potentias, et Extractio radicum: omnes aliae operationes, quae non sunt algebraicae, appellantur *transcendentes.* Hinc *functiones algebraicae* dicuntur, quae ex quantitatibus constantibus et variabilibus aut per solas operationes algebraicas, vel etiam per intermixtas transcendentes, quin tamen hae ipsas variabiles afficiant, inter se connexis componuntur: secus enim, si aliqua transcendens operatio afficiat quantitates variabiles, functiones quoque erunt *transcendentes*, inter quas functiones *trigonometricae*, *logarithmicae*, et *exponentiales* peculiarem merentur attentionem.

E. gr. Functiones ax, $x \operatorname{Sin} a$, $x \operatorname{Log} a$, x^a sunt algebraicae; et $a \operatorname{Sin} x$, $a \operatorname{Log} x$, a^x sunt functiones transcendentes.

Scholion I.

Palmaria principia trigonometrica, quorum amplissimus est usus per universam analysin mathematicam, quibusve etiam nos in sequentibus frequentissime utemur, sunt sequentia.

Assumto circuli radio $= 1$, positaque ejus semiperipheria $= \pi$; erit

1. $\pi = 3{,}141592653589793$ etc.
2. $\operatorname{Sin} 0\,\pi = \operatorname{Sin} \pi = 0$; $\operatorname{Sin} \tfrac{1}{2}\pi = 1$.

3. Cos

3. $\text{Cos}\, o\pi = 1$; $\text{Cos}\, \frac{1}{2}\pi = 0$; $\text{Cos}\, \pi = -1$.
4. $\text{Sin v}\, o\pi = 0$; $\text{Sin v}\, \frac{1}{2}\pi = 1$; $\text{Sin v}\, \pi = 2$.
5. $\text{Cos v}\, o\pi = \text{Cos v}\, \pi = 1$; $\text{Cos v}\, \frac{1}{2}\pi = 0$.
6. $\text{Tang}\, o\pi = \text{Tang}\, \pi = 0$; $\text{Tang}\, \frac{1}{2}\pi = \infty$.
7. $\text{Cot}\, o\pi = \infty$; $\text{Cot}\, \frac{1}{2}\pi = 0$; $\text{Cot}\, \pi = -\infty$.
8. $\text{Sec}\, o\pi = 1$; $\text{Sec}\, \frac{1}{2}\pi = \infty$; $\text{Sec}\, \pi = -1$.
9. $\text{Cosec}\, o\, \pi = \text{Cosec}\, \pi = \infty$; $\text{Cosec}\, \frac{1}{2}\pi = 1$.

Ceterum patebit ex sequentibus formulis, valorem cujuslibet functionis trigonometricae certi arcus per hujus sinum aut cosinum perfecte determinari: quodsi vero k denotet quemcunque terminum progressionis numericae o, 1, 2, 3, 4, 5, etc.; erit semper

10. $\text{Sin}\, 2k\pi = \text{Sin}\, (2k+1)\pi = 0$.
11. $\text{Cosin}\, 2k\pi = 1$; $\text{Cos}\, (2k+1)\pi = -1$.

Si retento radio $= 1$, dentur duo quicunque arcus, aut anguli a, b; erit

12. $\text{Sin}\, a^2 + \text{Cos}\, a^2 = 1$.
13. $\text{Sin}\, a = \sqrt{(1 - \text{Cos}\, a^2)} = \dfrac{\text{Tang}\, a}{\sqrt{(1 + \text{Tang}\, a^2)}}$.
14. $\text{Cos}\, a = \sqrt{(1 - \text{Sin}\, a^2)} = \dfrac{1}{\sqrt{(1 + \text{Tang}\, a^2)}}$.
15. $\text{Sin v}\, a = 1 - \text{Cosin}\, a$.
16. $\text{Cosin v}\, a = 1 - \text{Sin}\, a$.
17. $\text{Tang}\, a = \dfrac{\text{Sin}\, a}{\text{Cosin}\, a} = \dfrac{\text{Sin}\, a}{\sqrt{(1 - \text{Sin}\, a^2)}} = \dfrac{\sqrt{(1 - \text{Cosin}\, a^2)}}{\text{Cosin}\, a}$.
18. $\text{Cot}\, a = \dfrac{\text{Cosin}\, a}{\text{Sin}\, a} = \dfrac{1}{\text{Tang}\, a}$.
19. $\text{Sec}\, a = \dfrac{1}{\text{Cosin}\, a} = \dfrac{1}{\sqrt{(1 - \text{Sin}\, a^2)}}$.
20. $\text{Cosec}\, a = \dfrac{1}{\text{Sin}\, a} = \dfrac{1}{\sqrt{(1 - \text{Cosin}\, a^2)}}$.
21. $\text{Sin}\, (a+b) = \text{Sin}\, a\, \text{Cos}\, b + \text{Cos}\, a\, \text{Sin}\, b$.
22. $\text{Cos}\, (a+b) = \text{Cos}\, a\, \text{Cos}\, b - \text{Sin}\, a\, \text{Sin}\, b$.
23. $\text{Tang}\, (a+b) = \dfrac{\text{Tang}\, a + \text{Tang}\, b}{1 - \text{Tang}\, a\, \text{Tang}\, b}$.
24. $\text{Sin}\, (a-b) = \text{Sin}\, a\, \text{Cos}\, b - \text{Cos}\, a\, \text{Sin}\, b$.
25. $\text{Cos}\, (a-b) = \text{Cos}\, a\, \text{Cos}\, b + \text{Sin}\, a\, \text{Sin}\, b$.

26. $\text{Tang}(a-b)=\dfrac{\text{Tang } a-\text{Tang } b}{1+\text{Tang } a \text{ Tang } b}$.

27. $\text{Sin } 2a=2 \text{ Sin } a \text{ Cos } a=2 \text{ Sin } a\sqrt{(1-\text{Sin } a^2)}$.

28. $\text{Cosin } 2a=\text{Cos } a^2-\text{Sin } a^2=2 \text{ Cos } a^2-1=1-2 \text{ Sin } a^2$.

29. $\text{Tang } 2a=\dfrac{2 \text{ Tang } a}{1-\text{Tang } a^2}$.

30. $\text{Sin } \tfrac{1}{2} a=\sqrt{\dfrac{1-\text{Cosin } a}{2}}$.

31. $\text{Cos } \tfrac{1}{2} a=\sqrt{\dfrac{1+\text{Cosin } a}{2}}$.

32. $\text{Tang } \tfrac{1}{2} a=\sqrt{\dfrac{1-\text{Cosin } a}{1+\text{Cosin } a}}=\dfrac{\text{Sin } a}{1+\text{Cosin } a}$.

Sit h hyothenusa trianguli rectanguli; α, γ vero sint ejus catheti, et a, c anguli oppositi: erit

33. $h=\dfrac{\alpha}{\text{Sin } a}=\alpha \text{ Cosec } a$.

34. $\alpha=h \text{ Sin } a=\gamma \text{ Tang } a$.

35. $\gamma=\alpha\dfrac{\text{Cosin } a}{\text{Sin } a}=\alpha \text{ Cot } a$.

Porro sint a, b, c tria latera trianguli obliquanguli, et A, B, C, anguli iis oppositi: erit

36. $a=\sqrt{(b^2+c^2-2bc \text{ Cosin } A)}$.

37. $\text{Tang } B=\dfrac{b \text{ Sin } A}{c-b \text{ Cosin } A}$.

38. $\text{Cosin } A=\dfrac{b^2+c^2-a^2}{2bc}$.

Scholion 2.

Functiones hae (Schol. 1.) ad radium $=1$ relatae vocantur *naturales*, nimirum sinus naturales, cosinus naturales, et sic porro: relatae vero ad quemcunque alium radium r appellantur *artificiales*. Functiones naturales, quibus deinceps constanter utemur, convertentur, ubi opus fuerit, in artificiales, iisdem angulis, arcubusque similibus debitas, atque ad quemcunque datum radium r relatas, si illae multiplicentur per radium r. Sic etiam quivis arcus radio $=1$ descriptus convertetur in arcum similem descriptum radio r, si is ducatur in radium r.

E. gr.

E. gr. Sint a, b duo anguli, et functio $y = x \operatorname{Sin} a^2 - \sqrt{(1 - x \operatorname{Cos} b)}$ complectatur sinum naturalem anguli a, et cosinum naturalem anguli b: quodsi jam functio y determinari debeat per sinum artificialem anguli a, et cosinum artificialem anguli b, utrumque relatum ad datum radium r; poni debebit Sin art a = r Sin nat a, et Cos art b = r Cos nat b, hinc Sin nat a $= \frac{\text{Sin art a}}{r}$, et Cos nat b $= \frac{\text{Cos art b}}{r}$. His enim valoribus in y substitutis obtinebimus $y = \frac{x \operatorname{Sin} a^2}{r^2} - \sqrt{\left(1 - \frac{x \operatorname{Cos} b}{r}\right)}$, ubi Sin a et Cosin b non amplius naturalem, sed artificialem, ad radium r relatum, sinum anguli a, et cosinum anguli b denotabit.

Scholion 3.

Saepe fit, ut arcubus anguli, et vicissim angulis arcus in calculis analyticis substituantur, quod duplici modo potest praestari.

1. Si angulo recto = 90°. pro unitate assumto, φ denotet arcum e centro alicujus anguli α radio = 1 inter ejus crura descriptum; erit $\alpha = \frac{2\varphi}{\pi}$, et $\varphi = \frac{1}{2}\alpha\pi$ (1. Schol. 1. n.)

2. Quodsi autem angulus γ, pro quo arcus inter ejus crura radio = 1 descriptus aequetur eidem radio, pro unitate tanquam communi omnium angulorum mensura sumatur; poterit poni $\alpha = \varphi$, denotante α quemcunque angulum, et φ arcum inter ipsius crura radio = 1 descriptum; secus erit $\alpha = \gamma\varphi$, et $\varphi = \frac{\alpha}{\gamma}$.

6. Definitio.

Alia omnium functionum divisio est in *integras* et *fractas*. Ut aliqua functio sit *integra*, requiritur, ut ea nulli fractioni aequetur, in cujus denominatore occurrant quantitates variabiles elevatae ad potentias exponentium positivorum.

E. gr. Functiones bx^2; $\frac{c - x^3}{m}$; $\frac{e}{x^{-2}} = ex^2$ sunt integrae: contra fractae functiones sunt $\frac{a}{x} = ax^{-1}$; $\frac{a + bx}{c - x^2}$.

7. Definitio.

Functiones dividuntur porro in *rationales* et *irrationales*, prout illae nullam aut aliquam irrationalitatem, afficientem quantitates variabiles, complectuntur.

E. gr. Functiones $a x^2 = a x^{\frac{5}{3}}$; $\frac{a + x\sqrt{c}}{\sqrt{e - x^2}}$ sunt rationales: functiones vero $\frac{a - b\sqrt{x}}{c - x^2}$; $x^{\frac{1}{2}} = \sqrt{x}$ sunt irrationales.

8. Definitio.

Functio integra et rationalis (6. 7. §.) erit *primi*, vel *secundi*, aut *tertii*, vel generatim *mti Ordinis*, seu *unius*, *duarum*, *trium*, m *dimensionum*, prout maxima summa exponentium quantitatum variabilium in aliquo termino fuerit 1, vel 2, aut 3, vel generatim numerus integer m.

E. gr. Functio $a + x - y$ duarum variabilium x, y est 1*mi* ordinis seu unius dimensionis: functiones autem $a + bxy$, $c + dx^2 - ey^2$ sunt 2*di* ordinis seu duarum dimensionum: et functio $a + bx - cx^2 + exy^2$ est 3*tii* ordinis seu trium dimensionum.

9. Corollarium.

Functio integra et rationalis unius variabilis x est 1*mi*, 2*di*, 3*tii*, 4*ti*, m*ti* ordinis, si exponens maximae potestatis variabilis x est 1, vel 2, aut 3, vel 4, aut generatim numerus integer m (8. §.).

10. Definitio.

Functio fracta rationalis (6. 7. §.) est fractio *propria* seu *genuina*, aut *impropria:* prioris numerator est inferioris ordinis quam denominator: numerator vero posterioris ejusdem est aut altioris ordinis, quam denominator (8. 9. §.).

11. Corollarium.

Fractio impropria, si ejus numerator dividatur per denominatorem, resolvetur in duas partes, quarum una erit fractio propria, altera vero aut quantitas constans, aut functio integra, nisi fors numerator sit per denominatorem divisibilis, quo casu data functio fracta aut constanti cuipiam quantitati, aut integrae functioni aequabitur.

E. gr.

E. gr. $\frac{ax^2-2x}{bx^2-1}=\frac{a}{b}+\frac{\frac{a}{b}-2x}{bx^2-1}$.

$$\frac{x^5+2x^3-x+1}{x^3-3x}=x^2+5+\frac{14x+1}{x^3-3x}.$$

$$\frac{x^2-1}{x-1}=x+1.$$

12. DEFINITIO.

Numerus dimensionum in quovis termino datae functionis est summa exponentium, quibus quantitates variabiles in eodem termino sunt praeditae. Hinc repetenda est divisio functionum in *homogeneas* et *heterogeneas*. Functio integra vocatur *homogenea*, si idem est dimensionum numerus in quovis ejus termino: functio autem fracta erit *homogenea*, si numerator et denominator fuerint functiones homogeneae, ejusdem aut diversi ordinis (8. 9. §.).

13. DEFINITIO.

Denique notetur adhuc divisio functionum in *uniformes* et *multiformes*. Functio *uniformis* appellatur, quae pro quovis determinato valore quantitatis variabilis unum tantum valorem obtinet: *multiformis* vero functio vocatur, quae pro quolibet valore variabilis plures obtinet valores.

14. DEFINITIO.

Si variabilis absoluta x, ad quam functio y relata sit, crescere cogitetur, debebit ipsa functio y mutari (nimirum crescere vel decrescere): omne incrementum, quod variabilis x supponitur capere, vocatur *Differentia variabilis* x, cujus signum est Δx: quantitas vero, qua functio y mutatur (crescit nimirum vel decrescit), dum variabilis x augetur incremento Δx, dicitur Differentia functionis y, quae simili signo Δy deinceps exprimetur.

15. Corollarium 1.

Differentia Δx variabilis absolutae x potest denotare quamvis partem variabilis x, ita e. gr. ut inter omnes possibiles partes aliquotas $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, $\frac{1}{5}x$, etc. nulla sit, cui Δx non possit aequari (14. §.).

16. Corollarium 2.

Quantitas constans C (1. §.) nullam recipit differentiam; quocirca debebit poni $\Delta C=0$ (14. §.).

17. Corollarium 3.

Si y^1 denotet valorem, quem functio y relata ad variabilem absolutam x, hac in $x+\Delta x$ abeunte, obtineat; differentia ejusdem functionis erit $\Delta y = y^1 - y$, vel $\Delta y = y - y^1$, prout nimirum functio y, crescente variabili, pariter crescit vel decrescit (14. §.).

18. Corollarium 4.

Cum tamen sit $y - y^1 = -(y^1 - y)$, semper licebit differentiam Δy functionis y relatae ad variabilem absolutam x considerare instar incrementi, quod ipsa capit, dum variabilis x augetur incremento Δx, ita ut, si, x abeunte in $x+\Delta x$, y abeat in y^1, sit $\Delta y = y^1 - y$, modo differentia hac ratione determinata sumatur cum signis contrariis, utprimum functio y ejuscemodi esse deprehenditur, ut illa, crescente variabili x, debeat decrescere (17. §.).

19. Corollarium 5.

Quaelibet functio φ data per variabilem quamcunque censebitur valorem $\varphi^1 = \varphi + \Delta\varphi$ obtinere, si variabilis absoluta x, ad quam illa refertur (4. §.), augeatur incremento Δx (18. §.).

20. Corollarium 6.

Ex his elucet methodus generalis quaerendi expressionem differentiae Δy datae cujuscunque functionis y variabilis x: si enim ubique in y supponatur x abire in $x+\Delta x$, noteturque valor y^1, quem functio y hoc casu induat, et ab eo subtrahatur ipsa functio y; erit residuum aequale differentiae Δy (18. §.)

Exempla.

$$y = ax;\quad y^1 = a(x+\Delta x);$$
$$\Delta y = y^1 - y = ax + a\Delta x - ax = a\Delta x.$$
$$y = ax^2;\quad y^1 = a(x+\Delta x)^2;$$
$$\Delta y = y^1 - y = 2ax\Delta x + a\Delta x^2.$$
$$y = ax^3;\quad y^1 = a(x+\Delta x)^3$$
$$\Delta y = y^1 - y = 3ax^2\Delta x + 3ax\Delta x^2 + a\Delta x^3.$$

21. Corollarium. 7.

Quidquid sit functio Z variabilis absolutae x, quantitasque constans C, si x abeat in $x+\Delta x$; abibit functio $y = Z + C$ in $y^1 = Z + \Delta Z + C$, et $u = CZ$ abibit in $C(Z+\Delta Z) = u^1$ (19. §.): erit ergo $\Delta y = \Delta Z$, et $\Delta u = C\Delta Z$ (18. §.)

22. Co-

22. Corollarium 8.

Et si P, Q, R, --- Z sint quaecunque functiones variabilis absolutae x, debebit functio $v = P + Q + R + --- + Z$ obtinere aliquem valorem $y^I = P + \Delta P + Q + \Delta Q + R + \Delta R + --- + Z + \Delta Z$, si x augeatur incremento ΔX (19. §.): consequenter erit $\Delta y = \Delta P + \Delta Q + \Delta R + ---- + \Delta Z$ (18. §.).

23. Corollarium 9.

Duae functiones Z, X relatae ad unam variabilem absolutam x nequeunt dici aequales, nisi aequatio $Z = X$ pro quolibet valore variabilis x supponatur subsistere: quodsi ergo x abeat in $x + \Delta x$, quo casu functiones Z, X certos valores $Z + \Delta Z$, $X + \Delta X$ induent (19. §); debebit esse etiam $Z + \Delta Z = X + \Delta X$, hinc $\Delta Z = \Delta X$.

24. Theorema.

Si certo constet, quanta Z et X esse independentia a variabili z, esseque $Z = X + \alpha z^a + \beta z^b + \gamma z^c + \delta z^d +$ etc. *pro certis quantitatibus*, $\alpha, \beta, \gamma, ---$ a, b, c, etc. *pariter independentibus a z, et quovis valore ipsius variabilis z; erunt quanta Z, X inter se aequalia, et* $\alpha z^a + \beta z^b + \gamma z^c + \delta z^d +$ etc. $= 0$.

Demonstratio.

Clarum est, functionem $S = \alpha z^a + \beta z^b + \gamma z^c + \delta z^d +$ etc. si ea non est aequalis nihilo, diversos valores posse induere, pendentes a diversis valoribus quantitatis variabilis z, et quantitatibus $\alpha, \beta, \gamma, ---$ a, b, c, etc.: cum igitur quantum X, utpote independens a variabili z, eundem perpetuo valorem retinere censeri debeat, quomodocunque mutetur variabilis z; poterit quantum $Z = X + S$, si non est $S = 0$, diversos valores obtinere a certis valoribus variabilis z pendentes, quod pugnat cum assumta hypothesi. Eapropter debet esse $S = 0$, et $Z = X$.

25. Corollarium 1.

Impossibile est, ut pro quantis Z, X, P, Q, R, --- L, M, N, etc. independentibus a variabili z, et quovis valore hujus variabilis sit $Z + Pz + Qz^2 + Rz^3 +$ etc. $= X + Lz + Mz^2 + Nz^3 +$ etc., quin sit $Z = X$; nimirum $Z = X + (L - P)z + (M - Q)z^2 + (N - R)z^3 +$ etc. (24. §.).

26. Corollarium 2.

Functio $y = A + Bz^a + Cz^b + Dz^c +$ etc., in qua coefficientes et exponentes sint quantitates independentes a variabili z, non poterit esse

aequalis nihilo pro quovis valore variabilis z, quin fit eo ipso $A=o$ et $Bz^a+Cz^b+Dz^c+$ etc. $=o$ (24. §.).

27. Corollarium 3.

Si exponentes $a, b, c, d, - - - p, q, r, s$, etc. ordine crescant, certumque sit, pro quovis valore variabilis z tam functionem $y=A+Bz^a+Cz^b+ - - - +Pz^p+Qz^q+Rz^r+Sz^s+$ etc. quam aliquot primos coefficientes $A, B, C, - - - P$, seorsim sumtos aequari nihilo; erit eo ipso etiam $Qz^q+Rz^r+Sz^s+$ etc. $=o$, hinc $Q+Rz^{r-q}+Sz^{s-q}+$ etc. $=o$, et ideo etiam proxime sequens coefficiens $Q=o$ (26. §.).

28. Corollarium 4.

Quoties ergo constiterit, certam functionem $y=A+Bz^a+Cz^b+Dz^c+Ez^d+$ etc. pro quantitatibus $A, B, C, D, E, - - - a, b, c, d$, etc. independentibus a variabili z, et quovis valore ejusdem variabilis debere aequari nihilo; certum erit, singulos coefficientes A, B, C, D, etc. aequales esse nihilo (26. 27. §.).

29. Corollarium 5.

Et si binae functiones $P+Qz^a+Rz^b+Sz^c+$etc., $p+qz^a+rz^b+sz^c+$etc. pro quovis valore variabilis z inter se aequales sint; erunt etiam coefficientes potestatum aequalium variabilis z inter se aequales: nimirum erit $P-p+(Q-q)z^a+(R-r)z^b+(S-s)z^c+$etc. $=o$; hinc $P-p=o$, $Q-q=o$, $R-r=o$, $S-s=o$, etc. (28. §.); proinde $P=p$, $Q=q$, $R=r$, $S=s$, etc.

30. Corollarium 6.

His principiis innititur methodus quaerendi seriem formae $Az^a+Bz^b+Cz^c+$etc. aequalem datae functioni y variabilis z formam ab illa distinctam habenti. Si enim interea indeterminati coefficientes A, B, C, etc. et determinati exponentes a, b, c, etc., quales nimirum functio transformanda postulare visa fuerit, sumantur, assumtaque aequatio $y=Az^a+Bz^b+Cz^c+$etc. in aliam $U=V$ ita transformetur, ut utrumque membrum complectatur aliquam seriem illius formae, tum aut coaequentur coefficientes potestatum aequalium variabilis z contentarum in functionibus U, V, (29. §.), aut, translatis omnibus terminis ad unicum membrum, quo e. gr. fiat $U-V=o$, singuli coefficientes in $U-V$ ponantur aequales nihilo (28. §.); obtinebuntur hac ratione diversae aequationes inter coefficientes

inde-

indeterminatos A, B, C, etc. et quantitates constantes, a quibus pendet functio y, unde quaeri poterunt valores eorundem coefficientium.

31. Lemma.

Proprietas, quae tributa quantis A, B, C, D, - - - P *numero indeterminato* n *sumtis eo ipso etiam uno pluribus quantis* A, B, C, D, - - - P, Q, *debet convenire, conveniet quantis* A, B, C, D, - - - - P, Q, R, S, T, etc. *quovis numero sumtis, utprimum ea aliquot, e. gr. duobus tribusve, quantis convenerit.*

Demonstratio.

Si enim certum est, proprietatem, duobus tribusve quantis A, B, C, reipsa competentem, ejusmodi esse, ut, si ea generatim numero n quantis A, B, - - - P tribuatur, eo ipso etiam uno pluribus quantis A, B, - - - P, Q tribui debeat; debebit utique illa competere etiam quatuor quantis A, B, C, D, et ideo etiam quinque quantis A, B, C, D, E, hinc etiam sex quantis A, B, C, D, E, F; atque hac ratione licebit ex quovis quantorum numero ad numerum ipsorum unitate majorem concludere, quin sit possibile ad talem quantorum numerum n pervenire, ultra quem illam proprietatem non liceat extendere.

32. Theorema.

Si a, b, c, d, etc. *sint coefficientes independentes a variabili* z, *sitque* r *quiscunque numerus integer positivus; possibiles debebunt esse alii coefficientes* α, β, γ, δ, etc. *independentes a variabili* z, *pro quibus fieret* $(a+bz+cz^2+dz^3+\text{etc.})^r = a^r + r a^{r-1} bz + \alpha z^2 + \beta z^3 + \gamma z^4 + \text{etc.}$

Demonstratio.

1. Si M, N, O, - - - p, q, r, etc. sint quanta independentia a variabili z; possibiles erunt, prout id evidenter ex natura multiplicationis eluet, coefficientes α, β, γ, etc., pariter independentes a variabili z, pro quibus fieret

$$(M+Nz+Oz^2+\text{etc.})(p+qz+rz^2+\text{etc.}) =$$
$$= Mp+(Mq+Np)z+\alpha z^2+\beta z^3+\gamma z^4+\text{etc.}$$

2. Quare, sumta functione $Z = a+bz+cz^2+dz^3+\text{etc.}$, extabunt certi coefficientes α, β, γ, etc. independentes a z, pro quibus fieret $Z^2 = ZZ = a^2+2abz+\alpha z^2+\beta z^3+\gamma z^4+\text{etc.}$ ob (1).

3. Et

3. Et si pro quocunque indeterminato numero n, literis k, l, m, etc. quantitates independentes a variabili z denotantibus, ponatur $Z^n = a^n + n a^{n-1} bz + k z^2 + l z^3 + m z^4 + \text{etc.}$; possibiles debebunt esse alii coefficientes α, β, γ, etc. independentes a variabili z, pro quibus fieret $Z^{n+1} = Z^n . Z = a^{n+1} + (n+1) a^n bz + \alpha z^2 + \beta z^3 + \gamma z^4 + \text{etc.}$, ob (1.).

Hinc vero (2) (3) et (31. §.) evidenter sequitur, quod demonstrari debebat.

33. Corollarium 1.

Differentia cujuslibet functionis $u = A + Bv + Cv^2 + Dv^3 + Ev^4 + \text{etc.}$ variabilis v erit $\Delta y = A + B(v+\Delta v) + C(v+\Delta v)^2 + D(v+\Delta v)^3 + E(v+\Delta v)^4 + \text{etc.} - u$ (20. §.) $= (B + 2Cv + 3Dv^2 + 4Ev^3 + \text{etc.})\Delta v + \alpha\Delta v^2 + \beta\Delta v^3 + \gamma\Delta v^4 + \text{etc.}$, pro quibusdam coefficientibus α, β, γ, etc. independentibus a differentia Δv (32. §.).

34. Corollarium 2.

Datis quoque binis functionibus $\varphi = K + L\omega + M\omega^2 + N\omega^3 + \text{etc.}$ $\omega = pe + qe^2 + re^3 + se^4 + \text{etc.}$, possibiles erunt coefficientes α, β, γ, etc. independentes a quantitate e, pro quibus fieret $\varphi = K + Lpe + \alpha e^2 + \beta e^3 + \gamma e^4 + \text{etc.}$ (32. §.).

35. Problema.

Invenire seriem aequalem logarithmo functionis $u = 1 + z$ pertinenti ad systema indeterminatum.

Solutio.

1. Sint A, B, C, D, etc. coefficientes, pro quibus et quovis valore variabilis z debeat fieri.

$$\log u = Az + Bz^2 + Cz^3 + Dz^4 + - - - + Pz^r + Qz^{r+1}.$$

2. Extare debebunt coefficientes α, β, γ, δ, etc. independentes a differentia Δz variabilis z, pro quibus fieret (33. §.)

$$\Delta \log u = (A + 2Bz + 3Cz^2 + 4Dz^3 + - - - + rPz^{r-1} + (r+1)Qz^r)\Delta z + \alpha\Delta z^2 + \beta\Delta z^3 + \gamma\Delta z^4 + \delta\Delta z^5 + \text{etc.}$$

3. Est autem $\Delta \log u = \log u^1 - \log u$ (20. §.) $= \log(1 + z + \Delta z) - \log(1+z)$ (19. §.) $= \log\left(1 + \frac{\Delta z}{1+z}\right)$: ob (1.) deberet ergo esse etiam

$$\Delta \log u = \frac{A\Delta z}{1+z} + \frac{B\Delta z^2}{(1+z)^2} + \frac{C\Delta z^3}{(1+z)^3} + \frac{D\Delta z^4}{(1+z)^4} + \text{etc.}$$

4. Quam-

4. Quamobrem habebimus aequales series in (2) (3), quae, etiam divisae per Δz debebunt esse inter se aequales, pro quovis possibili valore differentiae Δz, quod est impossibile, quin per (25. §.) sit

$$A+2Bz+3Cz^2+4Dz^3+\cdots+rPz^{r-1}+(r+1)Qz^r=\frac{A}{1+z};$$

$$\text{hinc}\; \left.\begin{matrix}2B\\+A\end{matrix}\right)z \left.\begin{matrix}+3C\\+2B\end{matrix}\right)z^2 \left.\begin{matrix}+4D\\+3C\end{matrix}\right)z^3+\cdots+\left.\begin{matrix}(r+1)Q\\+rP\end{matrix}\right)z^r=0:$$

igitur per (28. §.)

$$\begin{array}{l|l} 2B+A=0 & B=-\frac{1}{2}A. \\ 3C+2B=0 & C=+\frac{1}{3}A. \\ 4D+3C=0 & D=-\frac{1}{4}A. \\ \vdots & \vdots \\ (r+1)Q+rP=0 & Q=-P.\frac{r}{r+1}. \end{array}$$

Adeoque series in (1) quaesita erit

$$ln + l(1+z)=A\left(z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+\cdots-\frac{z^{2n}}{2n}+\frac{z^{2n+1}}{2n+1}\right).$$

36. Corollarium 1.

Series in (35. §.) inventa semper subsistet, quemcunque valorem habeat z, positivum aut negativum: pro $-z$ loco z, erit ergo

$$l(1-z)=-A\left(z+\frac{1}{2}z^2+\frac{1}{3}z^3+\frac{1}{4}z^4+\cdots+\frac{z^{2n}}{2n}+\frac{z^{2n+1}}{2n+1}\right).$$

37. Corollarium 2.

Et quia est $l(1+z)-l(1-z)=l\left(\frac{1+z}{1-z}\right)$, obtinebimus ex (35. 36. §.) sequentem seriem:

$$l\left(\frac{1+z}{1-z}\right)=2A\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\frac{z^7}{7}+\cdots+\frac{z^{2n+1}}{2n+1}\right).$$

38. Corollarium 3.

Si b denotet basin systematis logarithmici, ponaturque $\frac{1+z}{1-z}=b$; erit $z=\frac{b-1}{b+1}$ et $l\left(\frac{1+z}{1-z}\right)=lb=1$: pro his valoribus obtinebitur ex (37. §.) sequens aequatio:

$$A=\frac{1}{2\left(\frac{b-1}{b+1}+\frac{(b-1)^3}{3(b+1)^3}+\frac{(b-1)^5}{5(b+1)^5}+\frac{(b-1)^7}{7(b+1)^7}+\text{etc.}\right)}.$$

39. Pro-

39. Problema.

Invenire seriem, quae numerum z dato logarithmo y in quocunque systemate respondentem exprimat.

Solutio.

1. Pro quovis logarithmo y et certis coefficientibus a, b, c, d, etc. independentibus ab y fit

$$z = 1 + ay + by^2 + cy^3 + dy^4 + --- + py^r + qy^{r+1}.$$

2. Erit differentia numeri z considerati instar unius functionis logarithmi variabilis y per (33. §.):

$$\Delta z = (a + 2by + 3cy^2 + 4dy^3 + --- + rPy^{r-1} + (r+1)qy^r)\Delta y$$
$$+ \alpha\Delta y^2 + \beta\Delta y^3 + \gamma\Delta y^4 + \delta\Delta y^5 + \text{etc.}$$

3. Cum autem sit in (2) $\Delta y = y^1 - y = lz^1 - lz = l(z + \Delta z) - lz$ $= l\left(1 + \frac{\Delta z}{z}\right)$; erit per (35. §.)

$$\Delta y = \frac{A\Delta z}{z} - \frac{A\Delta z^2}{2z^2} + \frac{A\Delta z^3}{3z^3} - \frac{A\Delta z^4}{4z^4} + \text{etc.}$$

4. Quamobrem debebunt dari certi coefficientes k, l, m, etc. independentes a differentia Δz, pro quibus et quovis valore ejusdem differentiae per (34. §.) in (2) (3) fieret

$$\Delta z = (a + 2by + 3cy^2 + 4dy^3 + --- + (r+1)qy^r)\frac{A}{z}\Delta z$$
$$+ k\Delta z^2 + l\Delta z^3 + m\Delta z^4 + n\Delta z^5 + \text{etc.}$$

et ideo etiam

$$z = Aa + 2Aby + 3Acy^2 + 4Ady^3 + --- + (r+1)Aqy^r$$
$$+ kz\Delta z + lz\Delta z^2 + mz\Delta z^3 + \text{etc.};$$

quod est impossibile, quin sit (24. §.)

$$z = Aa + 2Aby + 3Acy^2 + 4Ady^3 + --- + (r+1)Aqy^r.$$

Ob (1) per (29. §.) fiet ergo

$$\begin{array}{r|l}
Aa = 1 & a = \frac{1}{1.A}. \\
2Ab = a & b = \frac{1}{1.2A^2}. \\
3Ac = b & c = \frac{1}{1.2.3A^3}. \\
4Ad = c & d = \frac{1}{1.2.3.4A^4} \\
\vdots & \vdots \\
(r+1)Aq = p & q = p.\frac{1}{(r+1)A}.
\end{array}$$

Et

Et pro istis valoribus in (1) substitutis obtinebimus sequentem seriem pro numero z:

$$z = 1 + \frac{y}{A} + \frac{y^2}{2A^2} + \frac{y^3}{2.3.A^3} + \dots + \frac{y^r}{2.3.4\dots rA^r},$$

nimirum

$$z = 1 + \frac{lz}{A} + \frac{(lz)^2}{2A^2} + \frac{(lz)^3}{2.3A^3} + \frac{(lz)^4}{2.3.4A^4} + \dots + \frac{(lz)^r}{2.3.4\dots rA^r}.$$

40. Corollarium.

Si $z = b$ sit basis systematis, ad quod logarithmus $y = lz$ pertineat; erit $y = lz = lb = 1$: igitur in (39 §.)

$$b = 1 + \frac{1}{A} + \frac{1}{2A^2} + \frac{1}{2.3A^3} + \dots + \frac{1}{2.3.\dots rA^r}.$$

41. Definitio.

In quolibet systemate logarithmico datur, praeter ejus basim b, alter per ipsam determinatus numerus A (38. §.), per quem series $S = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 +$ etc. multiplicata generat productum aequale logarithmo functionis $1 + z$ (35. §.): numerum hunc vocant *Modulum* systematis, quo dato dabitur eo ipso etiam ejus basis (40. §.). *Logarithmi naturales* appellantur, quorum systema habet modulum unitati aequalem: omnes alii logarithmi ad quemcunque alium modulum, unitate majorem vel minorem, relati vocantur *artificiales*, inter quos praecipui sunt logarithmi *vulgares*, relati ad basim $= 10$.

42. Corollarium 1.

Denotante $b = e$ basin logarithmorum naturalium, erit, ob modulum $A = 1$ (41. §.), per (40. §.)

$$e = 2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \text{etc.} =$$
$$= 2{,}71828182 \text{ etc.}$$

43. Corollarium 2.

Pro basi $b = 10$ logarithmorum vulgarium (41. §.) reperietur illorum modulus A per (38. §.), et fractio $\frac{1}{A}$, quae seorsim notari meretur; erit enim:

$$A = \frac{1}{2\left(\frac{9}{11} + \frac{9^3}{3 \cdot 11^3} + \frac{9^5}{5 \cdot 11^5} + \frac{9^7}{7 \cdot 11^7} + \text{etc.}\right)} =$$

$$= 0,434294481903 \text{ etc.}$$

$$\frac{1}{A} = 2,302585092994 \text{ etc.}$$

44. Corollarium 3.

Si logarithmi $l(1+z)$, $L(1+z)$ unius numeri $1+z$ ad duo diversa systemata pertineant, quorum moduli sint m, M; erit $l(1+z) = mS$, $L(1+z) = MS$ (41. §.); hinc $l(1+z) : L(1+z) = m : M$: logarithmi cujusvis numeri pertinentes ad duo diversa systemata sunt ergo inter se in ratione directa modulorum eorundem systematum.

45. Corollarium 4.

Adeoque, si A denotet modulum logarithmorum vulgarium, ob modulum $= 1$ logarithmorum naturalium (41. §.), erit $1 : A = \log \text{nat } u : \log \text{vulg } u$ pro quovis numero u (44. §.): igitur fiet $l \text{ vulg } u = A\, l \text{ nat } u$, et $l \text{ nat } u = l \text{ vulg } u \cdot \frac{1}{A}$. Quare, cognitis logarithmis naturalibus certorum numerorum, reperientur logarithmi vulgares eorundem numerorum, si priores multiplicentur per modulum A logarithmorum vulgarium (43. §.); logarithmi autem vulgares quorumlibet numerorum ducti in fractionem $\frac{1}{A}$ (43. §.) dabunt logarithmos naturales eorundem numerorum.

46. Corollarium 5.

Si fiat $\frac{1+z}{1-z} = \frac{m^2}{m^2-1} = \frac{m^2}{(m+1)(m-1)}$; erit $z = \frac{1}{2m^2-1}$, et $l\left(\frac{1+z}{1-z}\right) = l\left(\frac{m^2}{(m+1)(m-1)}\right) = 2\,l\,m - (l(m+1) + l(m-1))$: quodsi jam hi valores substituantur in (37. §.), obtinebitur sequens admodum convergens series, pro logarithmis, quorum modulus est A (41. §.)

$$l\,m = \tfrac{1}{2}(l(m+1) + l(m-1))$$

$$+ A\left(\frac{1}{2m^2-1} + \frac{1}{3(2m^2-1)^3} + \frac{1}{5(2m^2-1)^5} + \frac{1}{7(2m^2-1)^7} + \text{etc.}\right).$$

47. Corollarium 6.

Quamobrem, posito modulo systematis $A = 1$ (41. §.), conversisque fractionibus $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{13}$, etc. in decimales derivabitur ex (46. §.) sequens formula pro logarithmis naturalibus.

$$\log \text{nat}\, m = \tfrac{1}{2}(\text{l nat}(m+1) + \text{l nat}(m-1))$$
$$+ \frac{1.\ 000000000000}{2m^2-1}$$
$$+ \frac{0.\ 333333333333}{(2m^2-1)^3}$$
$$+ \frac{0.\ 200000000000}{(2m^2-1)^5}$$
$$+ \frac{0.\ 142857142857}{(2m^2-1)^7}$$
$$+ \frac{0.\ 111111111111}{(2m^2-1)^9}$$
$$+ \frac{0.\ 090909090909}{(2m^2-1)^{11}}$$
$$+ \frac{0.\ 076923076923}{(2m^2-1)^{13}}$$
$$+ \text{etc.}\quad \text{etc.}$$

48. Corollarium 7.

Similem formulam pro logarithmo vulgari numeri m ex (46. §.) elicies, si, sumto modulo A per (43. §.), fractiones $\frac{A}{3}$, $\frac{A}{5}$, $\frac{A}{7}$, $\frac{A}{9}$, $\frac{A}{11}$, $\frac{A}{13}$, etc. in pure decimales convertas: erit enim

$$\text{l vulg}\, m = \tfrac{1}{2}(\text{l vulg}(m+1) + \text{l vulg}(m-1))$$
$$+ \frac{0.\ 434294481903}{2m^2-1}$$
$$+ \frac{0.\ 144764827301}{(2m^2-1)^3}$$
$$+ \frac{0.\ 086858896380}{(2m^2-1)^5}$$
$$+ \frac{0.\ 062042068843}{(2m^2-1)^7}$$

$$+\frac{0.0482549424833}{(2m^2-1)^9}$$
$$+\frac{0.0394813165526}{(2m^2-1)^{11}}$$
$$+\frac{0.0334072678838}{(2m^2-1)^{13}}$$
$$+ \text{etc. etc.}$$

Scholion.

Insignis est usus harum formularum in condendis tabulis logarithmicis, et corrigendis ampliandisque jam conditis. Si enim pro quocunque systemate quaerendi forent logarithmi numerorum naturalium 2, 3, 4, 5, 6, 7, et sic porro, ante omnia deberet determinari modulus A systematis, aut independenter ab ejus basi, aut per hanc datam (38. §.). Cognito autem modulo A, invenietur, posito $\frac{1+z}{1-z}=2$, hinc $x=\frac{1}{3}$, logarithmus binarii 2 per (37. §.): invento hoc logarithmo, logarithmi omnium aliorum numerorum per (46. §.) commodissime determinabuntur: sufficiet autem omni casu solos logarithmos numerorum primorum $m=3$, $m=5$, $m=7$, $m=11$, $m=13$, et ita porro, per (46. §.) quaerere, cum logarithmi numerorum compositorum 4, 6, 8, 9, 10, 12, etc. sola additione logarithmorum illorum factoribus debitorum possint determinari.

49. Problema.

Invenire seriem aequalem datae cuicunque quantitati exponentiali u^v.

Solutio.

Sit $z=u^v$; erit $lz=vlu$: sumtis ergo logarithmis naturalibus obtinebimus per (39. §.) ob $A=1$ (41. §.) sequentem seriem

$$u^v=1+vlu+\frac{(vlu)^2}{2}+\frac{(vlu)^3}{2.3}+\frac{(vlu)^4}{2.3.4}+\text{etc.}$$

50. Corollarium.

Pro basi $u=e$ logarithmorum naturalium erit $lu=le=1$; adeoque in (49. §.).

$$e^v=1+v+\frac{v^2}{2}+\frac{v^3}{2.3}+\frac{v^4}{2.3.4}+\frac{v^5}{2.3.4.5}+\text{etc.}$$

51. Pro-

51. Problema.

Invenire seriem aequalem potentiae $(1+x)^m = y$ *pro quovis exponente* m, *rationali et irrationali.*

Solutio.

1. Si m esset numerus integer positivus, series aequalis functioni y haberet formam $1+mx+\alpha x^2+\beta x^3+$ etc. (32. §.): ponamus idcirco dari coefficientes A, B, C, etc. pro quibus et quovis valore variabilis x, quolibet quoque alio exponente m fieri deberet.

$$y = 1+mx+Ax^2+Bx^3+\cdots+Qx^r+Rx^{r+1}.$$

2. Differentia hujus functionis debebit esse juxta (33. §.) pro quibusdam coefficientibus K, L, M, etc. independentibus a differentia Δx.

$$\Delta y = (m+2Ax+3Bx^2+\cdots+rQx^{r-1}+(r+1)Rx^r)\Delta x$$
$$+K\Delta x^2+L\Delta x^3+M\Delta x^4+ \text{etc.}$$

3. In (2) ob (1) erat $\Delta y = y^1 - y$ pro $y=(1+x)^m$, $y^1=(1+x+\Delta x)^m$. Est autem pro quovis possibili exponente m

$$\frac{y^1}{y} = \frac{(1+x+\Delta x)^m}{(1+x)^m} = \left(1+\frac{\Delta x}{1+x}\right)^m:$$

ob (1) debet ergo esse etiam

$$\frac{y^1}{y} = 1+\frac{m\Delta x}{1+x}+\frac{A\Delta x^2}{(1+x)^2}+\frac{B\Delta x^3}{(1+x)^3}+ \text{etc.};$$

et eo ipso

$$\Delta y = \frac{my}{1+x}\Delta x+\frac{Ay\Delta x^2}{(1+x)^2}+\frac{By\Delta x^3}{(1+x)^3}+ \text{etc.}$$

4. Quare debebunt esse series pro Δy in (2) (3) pro quovis valore differentiae Δx inter se aequales, etiam si illae per Δx dividantur, quod est impossibile, quin per (25. §.) sit

$$m+2Ax+3Bx^2+\cdots+rQx^{r-1}+(r+1)Rx^r = \frac{my}{1+x}.$$

Hinc demum et ex (1) obtinebimus

$$\left.\begin{matrix}2A\\+m\\-m^2\end{matrix}\right\}x \left.\begin{matrix}+3B\\+2A\\-mA\end{matrix}\right\}x^2 \left.\begin{matrix}+4C\\+3B\\-mB\end{matrix}\right\}x^3 \left.\begin{matrix}+\cdots+(r+1)R\\+\cdots\cdots+rQ\\+\cdots\cdots-mQ\end{matrix}\right\}x^r = 0.$$

 Unde

Unde per (28 §.) fiet

$$\begin{array}{l|l}
2A+m-m^2=0 & A=\frac{m(m-1)}{1\,.\,2}. \\
3B+2A-mA=0 & B=\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}. \\
4C+3B-mB=0 & C=\frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4}. \\
\vdots & \vdots \\
(r+1)R+rQ-mQ=0 & R=Q\,.\,\frac{m-r}{r+1}.
\end{array}$$

5. Valores hi in (1) fubftituti dabunt fequentem feriem, in qua terminus ultimus eft indeterminatus, ita ut ex illo finguli termini poft primum $=1$ ordine fequentes poffint derivari, fi fucceffive fiat $r=1$, $r=2$, $r=3$, et fic porro.

$$(1+x)^m=1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^3$$
$$+\;-\;-\;-\;+\frac{m(m-1)(m-2)\,---\,(m-r+1)}{1\,.\,2\,.\,3\,.\,----\,r}x^r.$$

52. Corollarium 1.

Cum fit $(a+b)^m=(1+x)^m.\,a^m$ pro $x=\frac{b}{a}$; prodibit pro his valoribus ex (51. §.) fequens formula exhibens potentiam indeterminatam cujuslibet binomii $a+b$.

$$(a+b)^m=a^m+\frac{m}{1}a^{m-1}b+\frac{m(m-1)}{1\,.\,2}a^{m-2}b^2$$
$$+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}b^3+\;-\;-\;-\;-\;-\;-\;-\;-$$
$$+\frac{m(m-1)(m-2)\,---\,(m-r+1)}{1\,.\,2\,.\,3\,----------\,r}a^{m-r}.b^r.$$

53. Corollarium 2.

Fiat $x=Q=\frac{b}{a}$ et $m=\frac{u}{v}$ in (51. §.); reperietur inde fequens formula, cujus ope nonnihil commodius poterit determinari potentia exponentis fracti $\frac{u}{v}$ dati binomii $a+b$, literis A, B, C, D, etc., ut per fe

clarum

clarum est, singulos terminos seriei, primum, secundum, tertium, quartum, etc., designantibus.

$$(a+b)^{\frac{u}{v}} = \underbrace{a^{\frac{u}{v}}}_{A} + \underbrace{\frac{u}{v}AQ}_{B} + \underbrace{\frac{u-v}{2v}BQ}_{C} + \underbrace{\frac{u-2v}{3v}CQ}_{D} + \text{etc.}$$

54. Corollarium 3.

Unum exemplum notatu dignum suppeditat functio $\sqrt{(ax^m+bx^n)} = (ax^m+bx^n)^{\frac{1}{2}}$: si ponas $Q = \frac{bx^n}{ax^m} = \frac{bx^{n-m}}{a}$, $u=1$, $v=2$ in (53. §.), invenies sequentem seriem:

$$\sqrt{(ax^m+bx^n)} = a^{\frac{1}{2}}x^{\frac{m}{2}} + \frac{bx^{\frac{2n-m}{2}}}{2a^{\frac{1}{2}}} - \frac{1.b^2x^{\frac{4n-3m}{2}}}{2.4a^{\frac{3}{2}}}$$

$$+ \frac{1.3.b^3x^{\frac{6n-5m}{2}}}{2.4.6.a^{\frac{5}{2}}} - \frac{1.3.5\,b^4.x^{\frac{8n-7m}{2}}}{2.4.6.8.a^{\frac{7}{2}}} + \text{---------}$$

$$\pm \frac{1.3.5\text{----}(2r-3)b^r.x^{\frac{2rn-(2r-1)m}{2}}}{2.4.6\text{----------}2ra^{\frac{2r-1}{2}}}.$$

55. Corollarium 4.

Aliud aeque memorabile exemplum praebet functio $\frac{1}{\sqrt{(ax^m+bx^n)}} = (ax^m+bx^n)^{-\frac{1}{2}}$. Si ponas $Q = \frac{bx^{n-m}}{a}$, $u=-1$, $v=2$; obtinebis ex (53. §.) sequentem seriem.

$$\frac{1}{\sqrt{(ax^m+bx^n)}} = \frac{1}{a^{\frac{1}{2}}.x^{\frac{m}{2}}} - \frac{1.b}{2a^{\frac{3}{2}}.x^{\frac{3m-2n}{2}}} + \frac{1.3\,b^2}{2.4a^{\frac{5}{2}}.x^{\frac{5m-4n}{2}}}$$

$$- \frac{1.3.5b^3}{2.4.6a^{\frac{7}{2}}.x^{\frac{7m-6n}{2}}} + \text{-------------}$$

$$\pm \frac{1.3.5\text{----------}(2r-1)b^r}{2.4.6\text{----}2ra^{\frac{2r+1}{2}}.x^{\frac{(2r+1)m-2rn}{2}}}.$$

56. Pro-

56. Problema.

Assumta serie Sin $\varphi = \varphi + b\varphi^2 + c\varphi^3 + d\varphi^4 + \ldots + p\varphi^{2n} + q\varphi^{2n+1}$, *invenire seriem pro* Cosin φ?

Solutio.

1. Si arcus variabilis φ crescat differentiâ $\Delta\varphi$; erit per (5. §. 1. Schol.)

$$\text{Sin}(\varphi + \Delta\varphi) = \text{Sin}\,\varphi\,\text{Cos}\,\Delta\varphi + \text{Cos}\,\varphi\,\text{Sin}\,\Delta\varphi.$$
$$\text{Cos}(\varphi + \Delta\varphi) = \text{Cos}\,\varphi\,\text{Cos}\,\Delta\varphi - \text{Sin}\,\varphi\,\text{Sin}\,\Delta\varphi.$$

2. Cum jam per hypothesin pro quovis arcu debeat esse Sin $\varphi = \varphi + b\varphi^2 + c\varphi^3 + d\varphi^4 + \ldots + p\varphi^{2n} + q\varphi^{2n+1}$:

3. Erit etiam Sin $\Delta\varphi = \Delta\varphi + b\,\Delta\varphi^2 + c\,\Delta\varphi^3 + d\,\Delta\varphi^4 + \ldots + q\,\Delta\varphi^{2n+1}$.

4. Porro est Cos $\Delta\varphi = (1 - \text{Sin}\,\Delta\varphi^2)^{\frac{1}{2}}$, unde evidenter elucet, Cos $\Delta\varphi$ aequari certae seriei, cujus terminus primus est unitas, reliqui vero termini post primum ordine sequentes aequales sunt productis ex Sin $\Delta\varphi^2$, Sin $\Delta\varphi^4$, Sin $\Delta\varphi^6$, etc. in quospiam coefficientes independentes a Sin $\Delta\varphi^2$ (52. §.): ob (3) per (34. §.) debebunt ergo dari aliqui coefficientes k, l, m, etc. independentes a $\Delta\varphi$, pro quibus fieret

$$\text{Cos}\,\Delta\varphi = 1 + k\,\Delta\varphi^2 + l\,\Delta\varphi^3 + m\,\Delta\varphi^4 + \text{etc.}$$

5. Differentia Δ Sin φ = Sin $(\varphi + \Delta\varphi)$ — Sin φ debebit esse ob (2) per (33. §.).

$$\Delta\,\text{Sin}\,\varphi = (1 + 2b\varphi + 3c\varphi^2 + 4d\varphi^3 + \ldots + 2np\varphi^{2n-1} + (2n+1)q\varphi^{2n})\,\Delta\varphi$$
$$+ \alpha\,\Delta\varphi^2 + \beta\,\Delta\varphi^3 + \gamma\,\Delta\varphi^4 + \delta\,\Delta\varphi^5 + \text{etc.}$$

6. Si autem sin $(\varphi + \Delta\varphi)$ determinetur per (1) (4) (3), tum subtrahatur Sin φ; obtinebitur pro differentia

$$\Delta\,\text{Sin}\,\varphi = \text{Cos}\,\varphi\,.\,\Delta\varphi + (k\,\text{Sin}\,\varphi + b\,\text{Cos}\,\varphi)\,\Delta\varphi^2 + (l\,\text{Sin}\,\varphi + c\,\text{Cos}\,\varphi)\,\Delta\varphi^3$$
$$+ (m\,\text{Sin}\,\varphi + d\,\text{Cos}\,\varphi)\,\Delta\varphi^4 + \text{etc.}$$

7. Quare, cum series (5) (6) inter se aequari debeant, divisaeque etiam per $\Delta\varphi$ semper sint futurae aequales, quidquid sit differentia $\Delta\varphi$; dabunt illae per (25. §.).

$$\text{Cos}\,\varphi = 1 + 2b\varphi + 3c\varphi^2 + 4d\varphi^3 + \ldots + 2np\varphi^{2n-1} + (2n+1)q\varphi^{2n}.$$

57. Cor-

57. Corollarium 1.

Ex hac porro ferie aliam pro Sin φ fimili modo licebit elicere. Si enim φ crefcat differentia $\Delta\varphi$, crefcet Cof φ differentia $\Delta \text{Cof}\,\varphi = \text{Cof}\,(\varphi + \Delta\varphi) - \text{Cof}\,\varphi$ (18. 20. §.), quae ob (56. §. 7. n) per (33. §.) erit

$$1.\ \Delta \text{Cof}\,\varphi = (2b + 2.3c\varphi + 3.4d\varphi^2 + \text{----} + (2n-1)\,2np\varphi^{2n-2} + 2n(2n+1)q\varphi^{2n-1} + \text{etc.})\,\Delta\varphi + \alpha\Delta\varphi^2 + \beta\Delta\varphi^3 + \gamma\Delta\varphi^4 + \text{etc.}$$

Et fi Cof $(\varphi + \Delta\varphi)$ per (56. §. n. 1. 4. 3.) determinetur; erit, fubtracto indidem Cof φ.

$$2.\ \Delta \text{Cof}\,\varphi = -\text{Sin}\,\varphi.\Delta\varphi + (k\,\text{Cof}\,\varphi - b\,\text{Sin}\,\varphi)\Delta\varphi^2 + (l\,\text{Cof}\,\varphi - c\,\text{Sin}\,\varphi)\Delta\varphi^3 + (m\,\text{Cof}\,\varphi - d\,\text{Sin}\,\varphi)\,\Delta\varphi^4 + \text{etc.}$$

Adeoque, quia feries (1) (2), etiam per $\Delta\varphi$ divifae, pro quavis poffibili differentia $\Delta\varphi$ inter fe aequari debent, erit per (25. §.) factor differentiae $\Delta\varphi$ in (1) aequalis factori $-$ Sin φ ejusdem differentiae in (2), proinde Sin $\varphi = -2b - 2.3c\varphi - 3.4d\varphi^2 - \text{---} - (2n-1)\,2np\varphi^{2n-2} - 2n(2n+1)q\varphi^{2n-1} - \text{etc.}$

58. Corollarium 2.

In feriebus aequalibus finum ejusdem arcus φ exprimentibus (56. 57. §.) deberet ergo effe $-2b = 0$, et $-(2n+1)(2n+2)r = p$, denotante r coefficientem potentiae φ^{2n+2} in ferie pro Sin φ (56. §.) per (29. §.): igitur effet $b = 0$, et $r = -p:(2n+1)(2n+2)$, unde neceffario fequitur, coefficientem b potentiae φ^2 in ferie pro Sin φ (56. §.) aequari nihilo, et coefficientem r cujuslibet potentiae φ^{2n+2} exponentis paris in eadem ferie aequari debere nihilo, fi coefficiens p potentiae φ^{2n} proxime inferioris exponentisque paris eft aequalis nihilo. Series ergo in (56. §.) affumta non poteft fubfiftere, nifi coefficientes omnium potentiarum φ^2, φ^4, φ^6, φ^8, et fic porro aequentur nihilo.

59. Problema.

Invenire feries aequales finui et cofinui dati cujuscunque arcus φ.

Solutio.

Sumtis interea indeterminatis coefficientibus C, E, - - - Q, S, etc. ponatur per (56. 58. 57. §).

$$1.\ \text{Sin}\,\varphi = \varphi + C\varphi^3 + E\varphi^5 + \text{----} + Q\varphi^{2n+1} + S\varphi^{2n+3}.$$

$$2.\ \text{Cof}\,\varphi = 1 + 3C\varphi^2 + 5E\varphi^4 + \text{---} + (2n+1)Q\varphi^{2n} + (2n+3)S\varphi^{2n+2}.$$

$$3.\ \text{Sin}\,\varphi = -2.3C\varphi - \text{----} - 2n(2n+1)Q\varphi^{2n-1} - (2n+2)(2n+3)S\varphi^{2n+1}.$$

In feriebus (1) (3) inter fe aequalibus erit per (29. §.) $-2.3C=1$, et generatim $-(2n+2)(2n+3)S=Q$; igitur

$$C=\frac{-1}{2.3}\text{; et } S=\frac{-Q}{(2n+2)(2n+3)}.$$

Hinc facile determinantur valores fingulorum coefficientium pro ferie (1), et eo ipfo etiam pro altera ferie (2): pro iis idcirco debebit effe

$$\operatorname{Sin}\varphi=\varphi-\frac{\varphi^3}{1.2.3}+\frac{\varphi^5}{1.2.3.4.5}-\frac{\varphi^7}{1.2\ldots7}+\cdots\pm\frac{\varphi^{2n+1}}{1.2.3\ldots(2n+1)}$$

$$\operatorname{Cof}\varphi=1-\frac{\varphi^2}{1.2}+\frac{\varphi^4}{1.2.3.4}-\frac{\varphi^6}{1.2\ldots6}+\cdots\pm\frac{\varphi^{2n}}{1.2.3\ldots2n}.$$

60. Problema.

Dato finu arcus φ invenire feriem aequalem ipfi arcui φ.

Solutio.

1. Sit $\operatorname{Sin}\varphi=z$; erit $\operatorname{Cof}\varphi=(1-z^2)^{\frac{1}{2}}$: per (52. §.) reperietur ergo fequens feries.

$$\frac{1}{\operatorname{Cof}\varphi}=(1-z^2)^{-\frac{1}{2}}=$$

$$=1+\frac{1.z^2}{2}+\frac{1.3\,z^4}{2.4}+\cdots+\frac{1.3.5\cdots(2r-1)z^{2r}}{2.4.6\cdots\cdots2r}.$$

2. Ponamus jam, debere fieri $\varphi=z+Az^2+Bz^3+Cz^4+Dz^5+Ez^6+\cdots+Pz^{2r-1}+Qz^{2r}+Rz^{2r+1}+Sz^{2r+2}+\text{etc.}$

3. Hinc, fi $\operatorname{Sin}\varphi=z$ crefcat differentia Δz, debebit arcus φ crefcere differentia (33. §.) $=$

$$=\Delta\varphi=(1+2Az+3Bz^2+4Cz^3+5Dz^4+6Ez^5+\cdots+(2r-1)Pz^{2r-2}+2rQz^{2r-1}+(2r+1)Rz^{2r}+(2r+2)Sz^{2r+1}+\text{etc.})\Delta z+\alpha\Delta z^2+\beta\Delta z^3+\gamma\Delta z^4+\text{etc.}$$

4. Sed eft, ob $z=\operatorname{Sin}\varphi$, $\Delta z=\operatorname{Sin}(\varphi+\Delta\varphi)-\operatorname{Sin}\varphi=\operatorname{Sin}\varphi\operatorname{Cof}\Delta\varphi+\operatorname{Cof}\varphi\operatorname{Sin}\Delta\varphi-\operatorname{Sin}\varphi$ (5. §. 1. Schol.): quodfi ergo $\operatorname{Cof}\Delta\varphi$, $\operatorname{Sin}\Delta\varphi$ exprimas per feries juxta (59. §.), invenies, pro certis coefficientibus k, l, m, etc. debere effe

$$\Delta z=\operatorname{Cof}\varphi.\Delta\varphi+k\Delta\varphi^2+l\Delta\varphi^3+m\Delta\varphi^4+\text{etc.}$$

5. Quamobrem in (3) (4) habebimus pro quibusdam coefficientibus K, L, M, etc. independentibus a $\Delta\varphi$ (34. §.)

$$\Delta\varphi=(1+2Az+\cdots\cdots+(2r+2)Sz^{2r+1})\operatorname{Cof}\varphi.\Delta\varphi+K\Delta\varphi^2+L\Delta\varphi^3+\text{etc.}$$

Hinc

Hinc, quia aequatio haec, etiam divisa per $\Delta\varphi$, pro quavis differentia $\Delta\varphi$ debet subsistere, necessario sequitur, unitatem aequalem factori ipsius $\Delta\varphi$ in primo membro aequari factori ejusdem differentiae in altero membro (24. §.), consequenter debere etiam esse

$$\frac{1}{\operatorname{Cos}\varphi} = 1 + 2Az + 3Bz^2 + 4Cz^3 + 5Dz^4 + 6Ez^5 + \text{-----} + (2r-1)Pz^{2r-2} + 2rQz^{2r-1} + (2r+1)Rz^{2r} + (2r+2)Sz^{2r+1}.$$

6. Hoc modo nascuntur binae inter se aequales series (1) (5), quarum idcirco coefficientes per (29. §.) coaequati dabunt

$$A=0;\ B=\frac{1}{2.3};\ C=0;\ D=\frac{1.3}{2.4.5};\ E=0;$$

$$Q=0;\ R=\frac{1.3.5\text{-----}(2r-1)}{2.4.6\text{----}2r(2r+1)}:$$

et pro his valoribus debebit esse in (2)

$$\varphi = z + \frac{1.z^3}{2.3} + \frac{1.3.z^5}{2.4.5} + \text{----} + \frac{1.3.\ldots(2r-1)z^{2r+1}}{2.4\text{---}2r(2r+1)}.$$

61. Theorema.

Quaelibet potentia polynomii $Az^a+Bz^b+Cz^c+Dz^d+$ etc. $=Z$, *in quo* A, B, C, D, --- a, b, c, d, etc. *sint quaecunque quantitates independentes a variabili* z, *aequatur alicui polynomio similis formae.*

Demonstratio.

1. Si binomium Az^a+Bz^b elevetur ad potentiam exponentis m, quisquis sit is, rationalis aut irrationalis, debebit haec potentia aequari certo polynomio formae in theoremate determinatae (52. §.).

2. Supponamus porro, quamvis potentiam polynomii $X=Az^a+Bz^b+Cz^c+\text{---}+Pz^p$ habentis terminos numero n aequari alicui polynomio similis formae.

3. Sumto adhuc uno termino Qz^q, habebimus per (52. §.) pro certis coefficientibus α, β, γ, etc. independentibus a z.

$$(X+Qz^q)^m = X^m + \alpha X^{m-1}.Qz^q + \beta X^{m-2}.Q^2z^{2q} + \gamma X^{m-3}.Q^3z^{3q} + \text{etc.}$$

Cum igitur singulae potentiae X^m, X^{m-1}, X^{m-2}, etc. aequentur polynomiis formae $Az^a+Bz^b+Cz^c+$ etc., ob (2); necesse est, ut eo ipso etiam potentia $(X+Qz^q)^m$ aequalis sit alicui polynomio similis formae.

Hinc jam (1) 2) 3) per (31. §.) evidenter sequitur, quod erat demonstrandum.

62. Corollarium.

Facile ex natura operationum algebraicarum colligitur, quotcunque functiones-formae $Az^a + Bz^b + Cz^c + Dz^d +$ etc. seu illae in unam summam addantur, seu subtrahantur ab invicem, seu inter se multiplicentur, aut dividantur, semper generare aliquam functionem similis formae: quare, cum praeterea omnis potentia talis functionis, quisquis sit ejus exponens, rationalis vel irrationalis, aequetur alicui functioni formae similis (61. §.), haud difficulter demonstratur, omnem functionem algebraicam unius variabilis z, fracta illa sit, vel integra, rationalis vel irrationalis, aut ex illis et his quomodocunque composita, aequalem esse alicui seriei formae $Az^a + Bz^b + Cz^c +$ etc.

63. Theorema.

Omnis functio, algebraica, logarithmica, trigonometrica, et ex his quacunque ratione composita, unius variabilis v *aequalis est certae seriei formae* $Av^a + Bv^b + Cv^c + Dv^d +$ etc.

Demonstratio.

1. Demonstravimus, sinum et cosinum cujuslibet arcus variabilis v aequalem esse certae functioni algebraicae ejusdem arcus, determinatae per seriem variarum potentiarum hujus arcus (59. 5. §.): cum igitur omnes reliquae functiones, ut Sinus versus, Cosinus versus, Tangens, Cotangens, Secans, Cosecans, certi arcus per Sinum et Cosinum ejusdem arcus perfecte possint determinari (5. §. 1. Schol.); certum est, omnes has functiones todidem functionibus algebraicis variabilis v aequari.

Ex iisdem principiis, ob (60. §.), patet eadem ratione, quemvis arcum variabilem v aequari posse certae functioni algebraicae sui sinus, vel cosinus, sinus versi, cosinus versi, tangentis, cotangentis, secantis, vel cosecantis.

2. Porro demonstratum est, logarithmum cujusvis variabilis $v = 1 + z$ aequari certae functioni algebraicae ejusdem variabilis (35. §.), et v aequari certae functioni algebraicae ipsius logarithmi lv spectati instar quantitatis variabilis (39. §.), quod deinde ad quantitates exponentiales extendimus (49. §.).

3. Ex his vero manifestum fit, omnem functionem logarithmicam, trigonometricam, et quocunque modo ex functionibus his et algebraicis compositam, aequari certae functioni algebraicae: consequenter debet quaevis functio

functio variabilis v, algebraica, trigonometrica, logarithmica, et quoquo modo ex his composita, aequari alicui seriei formae $Av^a + Bv^b + Cv^c + Dv^d +$ etc. quidquid sint coefficientes A, B, C, D, - - - et exponentes a, b, c, d, etc. independentes a v, et quisquis sit numerus terminorum ejus seriei (62. §.).

64. Corollarium 1.

Si y^I denotet valorem, quem functio $y = Av^a + Bv^b + Cv^c + Dv^d +$ etc. variabilis v indueret, si haec augeretur quantitate ω; erit $y^I = A(v+\omega)^a + B(v+\omega)^b + C(v+\omega)^c +$ etc.: igitur per (52. §.)

$$y^I = \left\{\begin{array}{l} Av^a + \frac{a}{1} Av^{a-1}.\omega + - - - + \frac{a(a-1) - - - (a-r+1)\,Av^{a-r}.\omega^r}{1.2 - - - - - - r} \\ + Bv^b + \frac{b}{1} Bv^{b-1}.\omega + - - - + \frac{b(b-1) - - - (b-r+1)\,Bv^{b-r}.\omega^r}{1.2 - - - - - - r} \\ + Cv^c + \frac{c}{1} Cv^{c-1}.\omega + - - - + \frac{c(c-1) - - - (c-r+1)\,Cv^{c-r}.\omega^r}{1.2 - - - - - - r} \\ + Dv^d + \frac{d}{1} Dv^{d-1}.\omega + - - - + \frac{d(d-1) - - - (d-r+1)\,Dv^{d-r}.\omega^r}{1.2 - - - - - - r} \\ + \quad \text{etc.} \quad \text{etc.} \quad - - - - \quad \text{etc.} \end{array}\right.$$

Haec igitur formula exhibet expressionem generalem valoris, quem data quaevis functio y variabilis v eo casu deberet obtinere, si haec quantitate ω augeretur, sicut series $Av^a + Bv^b + Cv^c + Dv^d +$ etc. generatim datae cuivis functioni y variabilis v potest aequari (63. §.).

65. Corollarium 2.

Cum autem prima series verticalis in (64. §.), eius terminis in unam summam collectis, exaequet ipsam functionem y, secunda vero series verticalis aequalis sit producto $\alpha\omega$ pro $\alpha = \frac{aAv^{a-1}}{1} + \frac{bBv^{b-1}}{1} + \frac{cCv^{c-1}}{1} + \frac{dDv^{d-1}}{1} +$ etc.; cumque tertia series verticalis sit indeterminata, ita ut ex illa omnes series verticales, post primam ordine sequentes possint determinari, si successive fiat $r=1$, $r=2$, $r=3$, $r=4$, etc.: evidens est, esse $y^I = y + \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 +$ etc. literis β, γ, δ, etc. ejusmodi quantitates, ab ω independentes denotantibus, ut quaevis illarum ex proxime praecedente eadem prorsus lege possit determinari, qua lege quantitas α determinatur ex ipsa functione y (64. §.).

 66. Co-

66. Corollarium 3.

Quidquid fit variabilis v, abfoluta, aut aliqua functio cujuspiam variabilis abfolutae x, neceffe eft, ut, dum variabilis abfoluta, ad quam functio y variabilis v relata fit, certa differentia augetur, v valorem aliquem $v + \Delta v$, et y certum valorem y^{I} obtineat (19. §.): ficut ergo quaevis data functio y variabilis v alicui feriei formae $Av^a + Bv^b + Cv^c + Dv^d +$ etc. aequatur (63. §.); fic quoque debet differentia Δy cujusvis datae functionis y variabilis v aequalis effe certae feriei formae $\alpha \Delta v + \beta \Delta v^2 + \gamma \Delta v^3 + \delta \Delta v^4 +$ etc., literis $\alpha, \beta, \gamma, \delta$, etc. quantitates independentes a Δv, et per ipfam functionem y ita determinatas defignantibus, ut fit $\alpha = \frac{aAv^{a-1}}{1} + \frac{bBv^{b-1}}{1} + \frac{cCv^{c-1}}{1} + \frac{dDv^{d-1}}{1} +$ etc., quaevis vero quantitatum $\alpha, \beta, \gamma, \delta$, etc. ex proxime praecedente, nimirum β ex α, γ ex β, δ ex γ, et fic porro, eadem prorfus lege poffit derivari, qua quantitas α derivatur ex functione y (64. 65. §.).

67. Corollarium 4.

Ceterum per fe liquet, feries, quarum uni functio y variabilis v, alteri vero ejus differentia Δy aequatur (66. §.), feu illae conftent terminis numero finitis, feu in infinitum excurrant, valoris effe finiti, quamdiu ipfa functio y, et differentia Δv variabilis v finita eft.

CAPUT II.
DE
NATURA ET INVENTIONE RATIONUM DIFFERENTIALIUM.

68. Definitio.

Quantum variabile Z crescens continuo aut decrescens *eo propius accedet* ad determinatum aliquem valorem V, quo minor fuerit differentia inter illud et hunc valorem: et si haec differentia possit evadere minor data quavis quantitate homogenea; erit V *Limes* seu *Terminus* quanti variabilis Z crescentis continuo vel decrescentis.

69. Definitio.

Ratio variabilis Z:X continuo crescens vel decrescens *eo propius accedet* ad aliquam determinatam rationem u:v, quo minor fuerit differentia inter illam et hanc rationem: erit autem ratio determinata u:v *Limes* seu *Terminus* rationis variabilis Z:X crescentis continuo vel decrescentis, si differentia inter has rationes possit evadere minor data quavis ratione.

70. Corollarium.

Ex (68. 69. §.) et natura rationum geometricarum facile colligitur differentiam inter rationem variabilem Z:X et rationem determinatam u:v posse fieri minorem data quavis ratione, si differentia inter exponentes $\frac{Z}{X}$, $\frac{u}{v}$ harum rationum fieri potest minor dato quovis exponente, adeoque rationem u:v exhibituram limitem rationis variabilis Z:X crescentis continuo vel decrescentis, si exponens prioris rationis fuerit limes exponentis rationis posterioris.

71. Theorema.

Dato quanto P *magnitudinis determinatae, et alio* U, *quod aequetur limiti quanti variabilis* Z *continuo crescentis vel decrescentis; erunt fractiones* $\frac{U}{P}$, $\frac{P}{U}$ *limites fractionum variabilium* $\frac{Z}{P}$, $\frac{P}{Z}$.

Demon-

Demonstratio.

Necesse est, ut extet differentia D, quae minor possit fieri dato quovis quanto, et pro qua semper sit $Z = U \pm D$ (68. §.), consequenter

$$\frac{Z}{P} = \frac{U}{P} \pm \frac{D}{P} \cdot \frac{P}{Z} = \frac{P}{U} \mp \frac{PD}{U^2 \pm UD}.$$

Casu primo est $\frac{D}{P}$ differentia inter $\frac{Z}{P}$ et $\frac{U}{P}$, eaque potest fieri minor data quavis quantitate u: cum enim per hypothesin possit fieri $D < Pu$; poterit quoque fieri $\frac{D}{P} < u$. Casu altero est $\frac{PD}{U^2 \pm UD}$ differentia inter $\frac{P}{Z}$ et $\frac{P}{U}$, eaque potest minor evadere data quacunque quantitate u: nam per hypothesin poterit fieri $D < \frac{U^2u}{P \mp uU}$; hinc etiam $PD \mp uUD < U^2u$, eoque ipso etiam $\frac{PD}{U^2 \pm UD} < u$.

72. Corollarium.

Si quantum U fuerit limes quanti variabilis Z crescentis continuo vel decrescentis, deturque quodcunque aliud quantum homogeneum P determinatae magnitudinis; erunt rationes determinatae $U:P$, $P:U$ limites rationum variabilium $Z:P$, $P:Z$ (71. 70. §.).

73. Theorema.

Si quanta U, V *exhibeant limites quantorum variabilium* Z, X, *crescentium continuo, vel decrescentium, seu ambo crescant simul, vel decrescant, aut crescente alterutro decrescat alterum; erit quotus* $\frac{U}{V}$ *limes quoti* $\frac{Z}{X}$.

Demonstratio.

Dari debent differentiae D, d, pro quibus semper sit $Z = U \pm D$, et $X = V \pm d$, ita ut differentiae D, d possint minores evadere datis quibuslibet quantitatibus homogeneis (68. §.): habebimus igitur

$$\frac{Z}{X} = \frac{U \pm D}{V \pm d} = \frac{U}{V} + \frac{\pm VD \mp Ud}{V^2 \pm Vd}.$$

Hinc autem elucet, differentiam inter $\frac{Z}{X}$ et $\frac{U}{V}$ seu sit prior quotus major, seu minor posteriore, generatim esse $= \frac{\pm VD \mp Ud}{V^2 \pm Vd}$, quae poterit fieri

fieri minor data quavis quantitate u. Cum enim possit evadere $D < \frac{1}{3} uV$, $d < \frac{1}{3}\frac{uV^2}{U}$, $d < \frac{1}{3}V$, adeoque $DV < \frac{1}{3}uV^2$, $dU < \frac{1}{3}uV^2$, $duV < \frac{1}{3}uV^2$; poterit etiam fieri $DV + dU + duV < uV^2$: igitur a fortiori generatim $\pm DV \mp dU \mp duV < uV^2$, hinc $\frac{\pm DV \mp dU}{V^2 \pm dV} < u$.

74. Corollarium.

Cognitis limitibus U, V duorum quantorum variabilium Z, X; erit ratio U:V limes rationis variabilis Z:X (73. 70. §.).

75. Theorema.

Datis quibuscunque quantis finitis P, Q, R, S, T, etc. *independentibus a quantitate variabili* ω, *possibilis erit valor hujus variabilis, pro quo fieret* $P\omega > Q\omega^2 + R\omega^3 + S\omega^4 + T\omega^5 +$ etc., *existente* P *quanto positivo, quidquid sint reliqua quanta.*

Demonstratio.

1. Constat, quantitatem variabilem ω in determinatione productorum $P\omega$, $Q\omega^2$, $R\omega^3$, etc. instar certi numeri considerari posse, qui nimirum aequetur exponenti rationis ejusdem quantitatis ad quantitatem pro ipsius mensura assumtam.

2. Porro certum est, quovis quanto finito majus esse possibile; et, datis duobus quibuscunque quantis $M > v$, possibilem esse numerum integrum n aequalem denominatori partis aliquotae $M\frac{1}{n}$ quanti majoris M, quae minor sit quanto v.

3. Iam vero pro positivo quanto P supponi possunt reliqua omnia quanta Q, R, S, T, etc. esse positiva: si enim in hac hypothesi fieri potest $P\omega > Q\omega^2 + R\omega^3 + S\omega^4 +$ etc.; poterit id a fortiori eo casu fieri, quo aliqua quantorum Q, R, S, T, etc. fuerint negativa. Quomodocunque autem dicantur crescere, vel decrescere, aut alterne crescere et decrescere quanta finita P, Q, R, S, T, etc.; semper erit possibile quantum Z majus singulis his quantis (2), pro quo idcirco fieret.

$$Z(\omega^2 + \omega^3 + \omega^4 + \text{etc.}) > Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.}$$

4. Pro quovis porro numero ω (1) est $Z\omega^2 > Z(\omega^2 + \omega^3 + \omega^4 + \text{etc.})$ $(1 - \omega)$, hinc $\frac{Z\omega^2}{1 - \omega} > Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.}$, ob (3).

 5. Cum

5. Cum igitur possibilis debeat esse numerus $\omega = \frac{1}{n}$, pro quo fieret $P > (P+Z)\omega$ ob (2), hinc $P\omega > \frac{Z\omega^2}{1-\omega}$; deberet esse pro eodem numero, etiam $P\omega > Q\omega^2 + R\omega^3 + S\omega^4 +$ etc., ob (4).

76. Corollarium 1.

Pro quibuslibet quantis finitis μ, M, N, O, etc. independentibus a variabili ω possibilis est valor hujus variabilis, pro quo summa $M\omega + N\omega^2 + O\omega^3 +$ etc. minor fieret dato quomodocunque parvo quanto μ, nimirum pro illo numero, pro quo esset $\mu\omega > M\omega^2 + N\omega^3 + O\omega^4 +$ etc. (75. §.).

77. Corollarium 2.

Assumto quoque indeterminato numero integro positivo r fieri poterit $P\omega^r > Q\omega^{r+1} + R\omega^{r+2} + S\omega^{r+3} +$ etc. pro certo valore variabilis ω (75. §.). Hinc autem facile perspicitur, cur sit impossibile, ut pro quantis K, L, M, N, etc. finitis independentibus a variabili ω, et pro quovis possibili valore hujus variabilis, summa $K\omega^r + L\omega^{r+1} + M\omega^{r+2} + N\omega^{r+3} +$ etc. sit valoris positivi aut negativi, quin primus terminus $K\omega^r$ habeat valorem positivum vel negativum.

78. Corollarium 3.

Functio $M\omega + N\omega^2 + O\omega^3 +$ etc. $= F$ est differentia inter U et U + F: quodsi ergo U, M, N, O, etc. sint quaecunque quanta finita independentia a variabili ω, quae continuo decrescere cogitetur; erunt quantum U et ratio U : 1 limites, ad quos functio U + F et ratio (U + F) : 1 eo propius accedent, quo magis decreverit ω (76. 72. §.).

Scholion 1.

Quod hic factum est, in sequentibus constanter observabitur, quanta nimirum, quae nonnunquam heterogenea esse videntur, inter se comparabuntur: sic generatim differentiae omnium functionum variabiliumque absolutarum inter se erunt comparabiles, licet eae quanta heterogenea, puta lineas, superficies, solida, tempora, celeritates, etc. exprimere videantur. Quaelibet nimirum functio y, ejus differentia Δy, et differentia Δx variabilis absolutae, ad quam illa refertur, censeri potest exhibere exponentem rationis geometricae quanti, cui ea deberet aequari, ad quantum pro ipsius mensura assumtum, quo ipso omnes expressiones pro y, Δy, Δx, ut numeri abstracti, redduntur inter se comparabiles.

Scho-

Scholion 2.

Pari prorsus ratione potest demonstrari, possibilem esse valorem variabilis ω, pro quo et dato quovis numero integro positivo m, quibuslibet quoque quantis positivis K, L, M, - - - U, V, fiat $\omega^m > K\omega^{m-1} + L\omega^{m-2} + M\omega^{m-3} + - - - U\omega + V$. Possibile est enim quantum Z majus singulis quantis K, L, M, - - - U, V seorsim sumtis, pro quo idcirco, posita summa $K\omega^{m-1} + L\omega^{m-2} + - - - - U\omega + V = S$, fieret $Z(\omega^{m-1} + \omega^{m-2} + \omega^{m-3} + - - - + \omega^1 + \omega^0) > S$: igitur etiam $\frac{Z(\omega^m - 1)}{\omega - 1} > S$, et a fortiori $\frac{Z\omega^m}{\omega - 1} > S$. Quare, cum possit fieri $\omega > Z + 1$, hinc $\omega^m > \frac{Z\omega^m}{\omega - 1}$: poterit etiam fieri $\omega^m > S$.

79. Definitio.

Sit y quaecunque functio variabilis absolutae x, et Δy ejus differentia debita differentiae Δx (14. §.). Possibile est, ut, differentia Δx continuo decrescente (15. §.), ratio $\Delta y : \Delta x$ eundem perpetuo valorem retineat, (e.gr. si sit $y = ax$, hinc $\Delta y = a\Delta x$, et $\Delta y : \Delta x = a : 1$): verum plerumque fiet, ut eo casu ratio $\Delta y : \Delta x$ continuo mutetur, crescat nimirum, vel decrescat. Ratio constans, cui ratio $\Delta y : \Delta x$ differentiae functionis y ad differentiam variabilis absolutae x, ad quam ea functio refertur (4. §.), perpetuo manet aequalis, vel, si haec ratio sit variabilis, limes, ad quem ratio $\Delta y : \Delta x$, differentia Δx continuo decrescente, propius semper et propius accesserit, vocabitur *Ratio differentialis* functionis y, cujus exponentem signo εy exprimemus. Methodus inveniendi rationes differentiales functionum est *Calculus differentialis.*

80. Corollarium 1.

Quantitas constans C nullius rationis differentialis est capax, quocirca ponetur hujus exponens $\varepsilon C = 0$ (16. 79. §.).

81. Corollarium 2.

Cum $y = x$ det $\Delta y = \Delta x$, hinc $\Delta y : \Delta x = 1 : 1$; patet, $1 : 1$ esse rationem differentialem variabilis absolutae x, ejusdemque exponentem $\varepsilon x = 1$ (79. §.).

82. Corollarium 3.

Si differentia alicujus functionis u relatae ad variabilem absolutam x ita dependeat a differentia Δx, ut pro certis quantitatibus finitis P, A, B, C, etc.

 inde-

independentibus a Δx fit $\Delta u = P\Delta x + A\Delta x^2 + B\Delta x^3 + C\Delta x^4 +$ etc. (66. 67. §.); hinc $\Delta u : \Delta x = (P + A\Delta x + B\Delta x^2 + C\Delta x^3 + \text{etc.}) : 1$; erit $P : 1$ ratio differentialis functionis u, et ejus exponens $\varepsilon u = P$ (79. 78. §.).

83. Theorema.

Cuivis functioni y relatae ad variabilem absolutam x debetur aliqua ratio differentialis, cujus exponens εy aequalis est quantitati finitae per ipsam functionem y certa lege determinatae.

Demonstratio.

Quidquid sit functio finita y variabilis absolutae x, necesse est, ut extent quaepiam quantitates α, β, γ, δ, etc pariter finitae per ipsam functionem y certa lege determinatae, pro quibus fieret (66. 67. §.) $\Delta y = \alpha\Delta x + \beta\Delta x^2 + \gamma\Delta x^3 + \delta\Delta x^4 +$ etc.: igitur $\varepsilon y = \alpha$ (82. §.).

84. Corollarium 1.

Quaelibet functio z valoris finiti relata ad variabilem absolutam x ita debet esse comparata, ut ejus differentia Δz pro quibusdam quantitatibus finitis k, l, m, etc. independentibus a Δx atque per ipsam functionem z perfecte determinatis aequetur seriei $\varepsilon z\Delta x + k\Delta x^2 + l\Delta x^3 + m\Delta x^4 +$ etc. (83. §.).

85. Corollarium 2.

Quamobrem, si εz fuerit valoris positivi vel negativi, indicio id erit, etiam $\Delta z = \varepsilon z\Delta x + k\Delta x^2 + l\Delta x^3 + m\Delta x^4 +$ etc. positivi esse vel negativi valoris (84. 77. §.): cum igitur sit $\Delta z = z^1 - z$, adeoque $z^1 > z$ vel $z^1 < z$ pro positivo vel negativo valore exponentis εz; necesse est, ut functio z, crescente variabili x, crescat vel decrescat, prout εz est positivi vel negativi valoris (20. §.).

86. Corollarium 3.

Pro $y = Ax^a + Bx^b + Cx^c +$ etc. esset $\alpha = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} +$ etc. et ideo etiam $\varepsilon y = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} +$ etc. (83. 66. §.)

87. Corollarium 4.

Sed etiam denotante v quamcunque functionem finitam variabilis absolutae x extabunt certae quantitates finitae P, Q, R, etc. independentes a Δv, pro quibus differentia cujuslibet functionis u datae per variabilem v fieret $\Delta u = P\Delta v + Q\Delta v^2 + R\Delta v^3 + S\Delta v^4 +$ etc. (66. 67. §.): cum igitur

Δv

Δv aequetur alicui seriei $\varepsilon v \Delta x + k \Delta x^2 + l \Delta x^3 +$ etc. (84. §.); patet, debere dari alios finitos coefficientes M, N, O, etc. independentes a Δx, pro quibus ob (34. §.) fieret $\Delta u = P \varepsilon v \Delta x + M \Delta x^2 + N \Delta x^3 + O \Delta x^4 +$ etc., et ideo $\varepsilon u = P \varepsilon v$ (82. §.).

88. Corollarium 5.

Pro $u = Av^a + Bv^b + Cv^c +$ etc. et quavis functione v variabilis absolutae x esset $P = aAv^{a-1} + bBv^{b-1} + cCv^{c-1} +$ etc., et $\varepsilon u = P \varepsilon v$ (87. 66. §.).

89. Corollarium 6.

Hinc generatim possumus statuere, exponentem rationis differentialis cujusvis functionis φ datae per variabilem z, si ejus differentia pro certis quantitatibus p, q, r, s. etc. independentibus a Δz sit $\Delta \varphi = p \Delta z + q \Delta z^2 + r \Delta z^3 + s \Delta z^4 +$ etc., debere esse $\varepsilon \varphi = p \varepsilon z$, ita ut exponens εz aut unitati, aut alicui quantitati finitae aequetur, prout est z aut variabilis absoluta (4. §.), aut certa functio alicujus variabilis absolutae (83. 87. §.).

90. Theorema.

Exponens rationis differentialis producti CZ *ex quacunque functione* Z *variabilis absolutae* x *in quantitatem constantem* C *aequatur producto ex exponente rationis differentialis functionis* Z *in eandem constantem* C, *seu est* $\varepsilon . CZ = C \varepsilon Z$.

Demonstratio.

Est enim $\Delta . CZ = C \Delta Z$ (21. §.) $= C \varepsilon Z \Delta x + Ck \Delta x^2 + Cl \Delta x^3 + Cm \Delta x^4 +$ etc. (84. §.): igitur $\varepsilon . CZ = C \varepsilon Z$ (82. §.).

91. Theorema.

Exponens rationis differentialis summae $Z + C$ *quantitatis constantis* C *et cujuscunque functionis* Z *relatae ad variabilem absolutam* x *aequatur exponenti rationis differentialis ejusdem functionis* Z, *seu est* $\varepsilon (Z + C) = \varepsilon Z$.

Demonstratio.

Cum sit $\Delta (Z + C) = \Delta Z$ (21. §.) $= \varepsilon Z \Delta x + k \Delta x^2 + l \Delta x^3 + m \Delta x^4 +$ etc. (84. §.); debet esse $\varepsilon (Z + C) = \varepsilon Z$ (82. §.).

92. Theorema.

Exponens rationis differentialis summae plurium functionum P, Q, R, --- Z *unius variabilis absolutae* x *aequatur summae exponentium rationum differentialium*

tialium singulis his functionibus debitarum, *seu est* $\varepsilon(P+Q+R+\cdots+Z) = \varepsilon P+\varepsilon Q+\varepsilon R+\cdots \varepsilon Z$.

Demonstratio.

Per (22. §.) fieret $\Delta(P+Q+R+\cdots+Z) = \Delta P+\Delta Q+\Delta R+\cdots+\Delta Z$: igitur pro certis quantitatibus K, L, M, etc. independentibus a Δx deberet fieri $\Delta(P+Q+R+\cdots+Z) = (\varepsilon P+\varepsilon Q+\varepsilon R+\cdots+\varepsilon Z)\Delta x+K\Delta x^2+L\Delta x^3+M\Delta x^4+$ etc. (84. §.), eoque ipso $\varepsilon(P+Q+\cdots+Z) = \varepsilon P+\varepsilon Q+\cdots+\varepsilon Z$ (82. §.).

93. Definitio.

Quam in (79. §.) definivimus, vocetur *prima* ratio differentialis functionis y relatae ad unam variabilem absolutam x: tum *secunda* ratio differentialis functionis y sit aequalis primae rationi differentiali exponentis primae rationis differentialis functionis y: et sic generatim quaevis m*ta* ratio differentialis functionis y aequetur primae rationi differentiali exponentis proxime inferioris, nimirum $(m-1)$*esimae*, rationis differentialis functionis y, ita ut, signis εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, $\cdots$ $\overset{r}{\varepsilon} y$, $\overset{r+1}{\varepsilon} y$ denotantibus exponentes rationum differentialium, primae, secundae, tertiae, et sic porro, functionis y, sit $\overset{2}{\varepsilon} y = \varepsilon.\varepsilon y$; $\overset{3}{\varepsilon} y = \varepsilon.\overset{2}{\varepsilon} y$; $\cdots$ $\overset{r+1}{\varepsilon} y = \varepsilon.\overset{r}{\varepsilon} y$.

94. Corollarium.

Concessa methodo inveniendi exponentem primae rationis differentialis cujusvis assignabilis functionis, habebitur eo ipso methodus determinandi exponentes rationum differentialium omnium ordinum (93. §.).

Scholion.

Attendat tyro, ne expressiones $\overset{n}{\varepsilon} y$, εy^n, $\varepsilon . y^n$ inter se confundat. Expressio enim $\overset{n}{\varepsilon} y$ denotat exponentem n*tae* rationis differentialis functionis y: εy^n autem tantundem ac $(\varepsilon y)^n$ significat, nimirum n*tam* potentiam exponentis εy primae rationis differentialis functionis y: et $\varepsilon . y^n$ idem, quod $\varepsilon(y^n)$, indigitat, nimirum exponentem primae rationis differentialis n*tae* potentiae functionis y. Quod autem ad signum ε generatim adtinet, censebitur id illam duntaxat totam quantitatem afficere, cui immediate fuerit praefixum, nullo interjecto signo, aut puncto interposito: sic e. gr. expressio $\varepsilon . z \varepsilon y$ denotabit exponentem primae rationis differentialis functionis per $z \varepsilon y$ expressae; contra $\varepsilon z\, \varepsilon y$ denotabit productum ex exponentibus εz, εy rationum differentialium debitarum functionibus z, y.

Sic

Sic quoque $\varepsilon.z\sqrt{(1-z^2)}$ exprimit exponentem rationis differentialis functionis $z\sqrt{(1-z^2)}$, et $\varepsilon z\sqrt{(1-z^2)}$ denotat productum ex exponente εz rationis differentialis functionis z in functionem $\sqrt{(1-z^2)}$.

95. Theorema.

Exponentes εy, εz *rationum differentialium functionibus aequalibus* $y=Y+A$, $z=Z+B$ *unius variabilis absolutae* x *debitarum debent esse inter se aequales.*

Demonstratio.

Quidquid sint constantes A, B; erit $\varepsilon y=\varepsilon Y$, $\varepsilon z=\varepsilon Z$ (91. §.), et $\Delta y=\Delta z$; si est $y=z$ (23. §.): pro certis coefficientibus k, l, m, - - - K, L, M, etc. independentibus a Δx; erit ergo $\varepsilon y\Delta x+k\Delta x^2+l\Delta x^3+m\Delta x^4+\text{etc.}=\varepsilon z\Delta x+K\Delta x^2+L\Delta x^3+M\Delta x^4+\text{etc.}$ (84. §.), quod est impossibile, quin sit $\varepsilon y=\varepsilon z$ (29. §.).

96. Theorema.

Si vero partes variabiles $Y>Z$ *duarum functionum* $y=Y+A$, $z=Z+B$ *unius variabilis absolutae* x *sint inaequales, quidquid sint partes constantes* A, B; *erit exponens rationis differentialis functionis* y *habentis majorem partem variabilem, major exponente rationis differentialis alterius functionis* z: *nimirum* $\varepsilon y>\varepsilon z$.

Demonstratio.

Suppono partem variabilem in quavis functione y et z a parte constanti perfecte esse separatam, ita ut e. gr. in functione y resoluta in seriem (63. §.) A denotet summam illorum duntaxat terminorum, qui carent variabili x, et Y exprimat summam reliquorum terminorum complectentium variabilem x. Sic si esset $y=Y+A=(a+x)^2+2b$, cum sit $(a+x)^2=a^2+2ax+x^2$; deberet esse pars variabilis $Y=2ax+x^2$, et pars constans $A=a^2+2b$.

Hoc praemisso manifestum fit, non posse esse $Y>Z$, nisi detur tertia aliqua functio X variabilis x, pro qua sit $Y=Z+X$: igitur $\varepsilon Y=\varepsilon Z+\varepsilon X$ (92. §.), hinc $\varepsilon Y>\varepsilon Z$, et $\varepsilon y>\varepsilon z$ (91. §.).

97. Corollarium 1.

Quoties exponentes εy, εz rationum differentialium debitarum duabus functionibus y, z unius variabilis absolutae x inter se aequales fuerint; toties debebunt esse etiam aut integrae functiones y, z, aut saltem illarum partes

partes variabiles inter se aequales (96. §.): eapropter est absolute impossibile, ut determinatus quispiam exponens rationis differentialis pluribus functionibus respondeat, quarum partes variabiles, non forma duntaxat, sed etiam magnitudine a se invicem differant.

98. Corollarium 2.

Si vero exponentes $sz > sy$ duarum rationum differentialium debitarum functionibus z, y unius variabilis absolutae x fuerint inaequales; erit quoque pars variabilis functionis z, cui major exponens respondet, major parte variabili alterius functionis y (95. 96. §.): et ideo est absolute impossibile, ut certa functio plures rationes differentiales recipiat, quarum exponentes, non tantum forma, sed et magnitudine a se invicem discrepent.

99. Definitio.

Si $y = Z$ sit aliqua functio variabilis z (e. gr. $y = 1 - z^2$), poterit quoque z spectari instar certae functionis $z = Y$ variabilis y, quae per aequationem $y = Z$ est determinata (in assumto exemplo erit ea functio $z = \sqrt{(1 - y)}$): determinatio functionis $z = Y$ variabilis y per functionem $y = Z$ variabilis z vocari potest *inversio functionis* y.

100. Corollarium.

Si data quaecunque functio $y = Z$ variabilis z inversa generet functionem $z = Y$ variabilis y; erit y respectu functionis $z = Y$ aut variabilis absoluta, aut aliqua functio cujuspiam variabilis absolutae x, prout fuerit z respectu functionis $y = Z$ vel variabilis absoluta, vel certa functio variabilis absolutae x (99. 4. §.): erit quoque functio $y = Z$ ita connexa cum functione $z = Y$, ut, dum pro quopiam valore v variabilis z ejus functio $y = Z$ certum valorem V induerit, valore V tributo ipsi y, tanquam quantitati variabili, ejus functio $z = Y$ valorem v debeat obtinere (99. §.).

101. Problema.

Si data quaecunque functio $y = Z$ *variabilis* z *invertatur, ut* $z = Y$ *fiat certa functio variabilis* y (99. §.); *determinare relationem inter exponentes rationum differentialium, datae functioni* $y = Z$ *variabilis* z, *et alteri per illius inversionem obtentae functioni* $z = Y$ *variabilis* y, *debitarum.*

Solutio.

Solutio.

1. Sit $y = Az^a + Bz^b + Cz^c +$ etc., unde per inversionem (99. §.) nascatur $z = Py^p + Qy^q + Ry^r +$ etc. (63. §).

2. Erit $\varepsilon y = (aAz^{a-1} + bBx^{b-1} + cCz^{c-1} +$ etc.$)\,\varepsilon z$, et $\varepsilon z = (pPy^{p-1} + qQy^{q-1} + rRy^{r-1} +$ etc.$)\,\varepsilon y$, ubi erit $\varepsilon z = 1$ in prima, et $\varepsilon y = 1$ in secunda expressione, si z in functione y (1), adeoque y in functione z (1) fuerit variabilis absoluta (100. 86 88 §.).

3. Ponamus autem, assumta utcunque parva quantitate e, variabilem y in functione z (1) abire in $y + e$, ipsam idcirco functionem z in (1) capere incrementum $E = (pPy^{p-1} + qQy^{q-1} + rRy^{r-1} +$ etc.$)e + ke^2 + le^3 + me^4 +$ etc. (65. §.).

4. Quodsi jam $z + E$ sumatur loco z in functione y (1); debebit functio y in (1) ob (100. §.) capere incrementum e, nimirum abire in $y + e$, sicut z in (3) supponitur abire in $z + E$: incrementum hoc e functionis y in (1) erit ergo per (65. §.) $e = (aAz^{a-1} + bBz^{b-1} + cCz^{c-1} +$ etc.$)\,E + \beta E^2 + \gamma E^3 +$ etc.

5. Ex (3) per (32. §.) elucet, quamvis *ntam* potentiam incrementi E aequari alicui polynomio formae generalis $\varphi^n e^n + Ke^{n+1} + Le^{n+2} +$ etc., pro $\varphi = pPy^{p-1} + qQy^{q-1} + rRy^{r-1} +$ etc.: si igitur potentiae E, E^2, E^3, E^4, etc. per ejusmodi polynomia determinatae in (4) substitui cogitentur; debebit inde pro certis coefficientibus f, g, h, etc. independentibus ab e nasci aequatio $e = (aAz^{a-1} + bBz^{b-1} + cCz^{c-1} +$ etc.$)(pPy^{p-1} + qQy^{q-1} + rRy^{r-1} +$ etc.$)\,e + fe^2 + ge^3 + he^4 +$ etc., quae per e divisa dabit ob (24. §.) $1 = (aAz^{a-1} + bBz^{b-1} + cCz^{c-1} +$ etc.$)\,(pPy^{p-1} + qQy^{q-1} + rRy^{r-1} +$ etc.), unde et ex (2) elucet quaesita relatio.

102. Corollarium.

Quoties fuerit $\varepsilon y = \alpha \varepsilon z$ exponens rationis differentialis functionis y datae per variabilem z; toties exprimet $\varepsilon z = \frac{\varepsilon y}{\alpha}$ exponentem rationis differentialis functionis z variabilis y, quae per inversionem ex data functione y variabilis z potest obtineri, ita ut, si fuerit $\varepsilon z = 1$ in $\varepsilon y = \alpha \varepsilon z$ ob functionem y relatam ad variabilem absolutam z, debeat etiam esse $\varepsilon y = 1$ in $\varepsilon z = \frac{\varepsilon y}{\alpha}$ ob functionem z relatam ad variabilem absolutam y (100. 101. §.).

103. Definitio.

Differentiatio datae functionis y est determinatio illius rationis differentialis dependens ab inventione exponentis εy (79. §.), quem deinceps nomine *exponentis differentialis* intelligemus.

104. Problema.

Differentiare functionem formae $y = Az^a + Bz^b + Cz^c +$ etc. *pro quibuslibet coefficientibus* A, B, C, etc. *et exponentibus* a, b, c, etc. *rationalibus et irrationalibus, quavis item variabili* z, *absoluta sit, aut aliqua functio certae variabilis absolutae* x. (4. §.).

Solutio.

Multiplicentur singuli termini per exponentes potentiarum variabilis z in iisdem contentarum, ipsis autem potentiis substituantur potentiae uno gradu inferiores, tum totum ducatur adhuc in exponentem εz rationis differentialis variabilis z; erit quaesitus exponens differentialis (103. §.) $\varepsilon y = (aAz^{a-1} + bBz^{b-1} + cCz^{c-1} +$ etc.$)\,\varepsilon z$, pro $\varepsilon z = 1$, si z est variabilis absoluta (86. 88. §.).

105. Problema.

Differentiare functionem $y = ZX$ *aequalem producto ex duabus functionibus* Z, X *unius variabilis absolutae* x.

Solutio.

Multiplicetur exponens differentialis cujusvis factoris seorsim sumti per factorem alterum; erit summa productorum aequalis quaesito exponenti differentiali εy (103. §.), nimirum $Z\varepsilon X + X\varepsilon Z = \varepsilon y$.

Demonstratio.

Si variabilis absoluta x, ad quam functiones Z, X sunt relatae, augeatur differentia Δx; abibit functio $y = ZX$ in $y^1 = (Z + \Delta Z)(X + \Delta X)$, eritque $\Delta y = Z\Delta X + X\Delta Z + \Delta X \Delta Z$ (19 20. §.). Si ergo sumantur series differentiis ΔX, ΔZ aequales per (84 §.); debebit pro certis coefficientibus α, β, γ, etc. independentibus a Δx fieri $\Delta y = (Z\varepsilon X + X\varepsilon Z)\Delta x + \alpha\Delta x^2 + \beta\Delta x^3 + \gamma\Delta x^4 +$ etc., hinc $\varepsilon y = Z\varepsilon X + X\varepsilon Z$ (82. §.).

106. Corollarium 1.

Si functio $u = PQR - - - U$ aequetur producto ex functionibus P, Q, R, - - - U numero n, et $y = PQR - - - UV$ sit productum ex functionibus

ctionibus numero $n+1$; erit $\epsilon y = \epsilon . PQR --- UV = PQR --- U \epsilon V + V \epsilon PQR --- U$ (105. §.): quare, si $\epsilon . P, Q, R, --- U$ supponatur aequari summae productorum ex exponentibus differentialibus singulorum factorum $P, Q, R, --- U$ in omnes reliquos factores, debebit ϵy aequari summae productorum, quae nascerentur, si exponens differentialis cujusvis functionis $P, Q, R, --- U, V$ seorsim duceretur in omnes reliquas functiones.

107. Corollarium 2.

Hinc (106. §.) et ex (105. §.) per (31. §.) sequitur generatim, exponentem differentialem producti ex quotcunque functionibus aequalem esse summae productorum, quae obtinerentur, si exponens differentialis cujuslibet functionis seorsim sumtae multiplicaretur per omnes reliquas functiones.

108. Problema.

Differentiare functionem fractam $y = \frac{U}{V}$ *relatam ad unam variabilem absolutam* x.

Solutio.

Subtrahatur productum ex numeratore in exponentem differentialem denominatoris a producto ex denominatore in exponentem differentialem numeratoris, et residuum dividatur per quadratum denominatoris; erit quotus aequalis quaesito exponenti differentiali (103. §.), nimirum $\epsilon y = \frac{V \epsilon U - U \epsilon V}{V^2}$.

Demonstratio.

Ob $yV = U$ erit $y \epsilon V + V \epsilon y = \epsilon U$ (95. 105. §.), unde facile elucet, quod erat demonstrandum.

109. Problema.

Differentiare logarithmum $y = lu$ *spectatum instar unius functionis variabilis* u, *absoluta ea sit, aut aequalis cuicunque functioni alicujus variabilis absolutae* x.

Solutio.

Capto exponente differentiali variabilis u, dividatur is per ipsam variabilem u, et quotus ducatur in modulum M systematis, ad quod logarithmus $y = lu$ spectat; erit productum aequale exponenti differentiali loga-

rithmi $y = lu$, qui petebatur (103. §.) nimirum $\varepsilon y = \frac{\varepsilon u}{u} M$, vel $\varepsilon y = \frac{M}{u}$, si u est variabilis absoluta.

Demonstratio.

In (35. §. 3. n.) per (41. §.), sumto modulo M pro A, et posita variabili $u = 1 + z$, hinc $\Delta u = \Delta z$ (21. §.), erit pro numeris ibidem determinatis B, C, D, etc., $\Delta lu = \frac{M \Delta u}{u} + \frac{B \Delta u^2}{u^2} + \frac{C \Delta u^3}{u^3} +$ etc.; hinc $\varepsilon . lu = \frac{M \varepsilon u}{u}$ (89. §.).

110. Corollarium 1.

Pro logarithmis naturalibus, quibus deinceps constanter utemur, est modulus $M = 1$ (41. §.): igitur $\varepsilon y = \varepsilon lu = \frac{\varepsilon u}{u}$, vel $\varepsilon lu = \frac{1}{u}$, si u est variabilis absoluta (109. §.).

111. Corollarium 2.

Si dati quicunque logarithmi spectentur instar quantitatum variabilium; poterunt per (110. §.) etiam logarithmi logarithmorum facile differentiari. Sic e. gr. $y = llu$ dabit $\varepsilon y = \frac{\varepsilon lu}{lu} = \frac{\varepsilon u}{u lu}$ (110. §.); et $y = lllu$ dabit $\varepsilon y = \frac{\varepsilon . llu}{llu} = \frac{\varepsilon u}{ulu . llu}$; et ita porro.

112. Corollarium 3.

Quantitates vero exponentiales differentiabuntur per eadem praecepta, si capiantur illarum logarithmi naturales. Sic functio $y = a^z$ dat $ly = zla$, hinc $\frac{\varepsilon y}{y} = \varepsilon z la$ (95. 110. 90. §.), adeoque $\varepsilon y = a^z \varepsilon z la$. Et si in $y = v^z$ sint v, z duae quaecunque functiones relatae ad unam variabilem absolutam x; erit $ly = zlv$, et $\frac{\varepsilon y}{y} = z \frac{\varepsilon v}{v} + \varepsilon z lv$ (105. 110. §.), hinc $\varepsilon y = z v^{z-1} . \varepsilon v + v^z \varepsilon z lv$.

113. Corollarium 4.

Pro basi e logarithmorum naturalium est $le = 1$; hinc $\varepsilon . e^z = e^z \varepsilon z$ (112. §.).

114. Corollarium 5.

Cognito autem exponente differentiali logarithmi naturalis datae cujuscunque variabilis u, si is ducatur in modulum aut logarithmorum vulgarium (43. §.), aut aliorum logarithmorum artificialium, obtinebitur exponens differentialis logarithmi vulgaris, aut alterius logarithmi artificialis ejusdem variabilis u (109. 110. §.).

115. Problema.

Consideratis sinu et cosinu arcus variabilis φ instar duarum functionum ejusdem arcus, invenire exponentes rationum differentialium iis functionibus debitarum.

Solutio.

Functiones ejusmodi habentur in (59. §.), et illarum differentiae in (56. §. 6. n. 57. §. 2. n.): nimirum pro certis coefficientibus M, N, O, - - - p, q, r, etc. independentibus a $\Delta\varphi$ debet esse $\Delta \operatorname{Sin}\varphi = \Delta\varphi \operatorname{Cos}\varphi + M\Delta\varphi^2$ $N\Delta\varphi^3 + O\Delta\varphi^4 +$ etc., et $\Delta \operatorname{Cos}\varphi = -\Delta\varphi \operatorname{Sin}\varphi + p\Delta\varphi^2 + q\Delta\varphi^3 + r\Delta\varphi^4$ + etc. Quidquid sit ergo φ, variabilis absoluta, aut functio quaecunque alicujus variabilis absolutae; erit per (89. §.).

$$\varepsilon \operatorname{Sin}\varphi = \varepsilon\varphi \operatorname{Cos}\varphi. \quad \varepsilon \operatorname{Cos} = -\varepsilon\varphi \operatorname{Sin}\varphi.$$

116. Corollarium 1.

Cum sit $\operatorname{Sin} v\varphi = 1 - \operatorname{Cos}\varphi$ et $\operatorname{Cos} v\varphi = 1 - \operatorname{Sin}\varphi$; obtinebimus per (91. 115. §.) sequentes exponentes differentiales:

$$\varepsilon \operatorname{Sin} v\varphi = \varepsilon\varphi \operatorname{Sin}\varphi. \quad \varepsilon \operatorname{Cos} v\varphi = -\varepsilon\varphi \operatorname{Cos}\varphi.$$

117. Corollarium 2.

Est porro $\operatorname{Tang}\varphi = \frac{\operatorname{Sin}\varphi}{\operatorname{Cos}\varphi}$, et $\operatorname{Cot}\varphi = \frac{\operatorname{Cos}\varphi}{\operatorname{Sin}\varphi}$: per (108. 115. §.) debebit igitur esse.

$$\varepsilon \operatorname{Tang}\varphi = \frac{\varepsilon\varphi}{\operatorname{Cosin}\varphi^2} = \varepsilon\varphi \operatorname{Sec}\varphi^2.$$

$$\varepsilon \operatorname{Cot}\varphi = \frac{-\varepsilon\varphi}{\operatorname{Sin}\varphi^2} = -\varepsilon\varphi \operatorname{Cosec}\varphi^2.$$

118. Corollarium 3.

Denique est $\operatorname{Sec}\varphi = \operatorname{Cos}\varphi^{-1}$, et $\operatorname{Cosec}\varphi = \operatorname{Sin}\varphi^{-1}$: per (104. 115. §.) inveniemus ergo

$$\varepsilon \operatorname{Sec}\varphi = \frac{\varepsilon\varphi \operatorname{Sin}\varphi}{\operatorname{Cosin}\varphi^2} = \varepsilon\varphi \operatorname{Tang}\varphi \operatorname{Sec}\varphi.$$

$$\varepsilon \operatorname{Cosec}\varphi = \frac{-\varepsilon\varphi \operatorname{Cos}\varphi}{\operatorname{Sin}\varphi^2} = -\varepsilon\varphi \operatorname{Cot}\varphi \operatorname{Cosec}\varphi.$$

119. Corollarium 4.

Hactenus spectavimus sinus, cosinus, sinus et cosinus versos, tangentes, cotangentes, secantes, et cosecantes arcuum variabilium φ instar totidem functionum horum arcuum, quo casu erit ubique in praecedentibus $\varepsilon\varphi = 1$, si φ nulli functioni alicujus variabilis absolutae aequetur, sed sit ipsa variabilis absoluta: si ergo illae functiones cogitentur inverti, ut ex iis nascantur functiones exprimentes arcus φ per ipsorum sinus, cosinus, etc. (99. §.); obtinebimus pro his novis functionibus per (102. §.) ex (115. 116. 117. 118. §.) sequentes exponentes differentiales:

$$1.\ \varepsilon\varphi = \frac{\varepsilon \operatorname{Sin}\varphi}{\operatorname{Cosin}\varphi}. \qquad 2.\ \varepsilon\varphi = -\frac{\varepsilon \operatorname{Cos}\varphi}{\operatorname{Sin}\varphi}.$$

$$3.\ \varepsilon\varphi = \frac{\varepsilon \operatorname{Sin} v\,\varphi}{\operatorname{Sin}\varphi}. \qquad 4.\ \varepsilon\varphi = -\frac{\varepsilon \operatorname{Cos} v\,\varphi}{\operatorname{Cosin}\varphi}.$$

$$5.\ \varepsilon\varphi = \frac{\varepsilon \operatorname{Tang}\varphi}{\operatorname{Sec}\varphi^2}. \qquad 6.\ \varepsilon\varphi = -\frac{\varepsilon \operatorname{Cot}\varphi}{\operatorname{Cosec}\varphi^2}.$$

$$7.\ \varepsilon\varphi = \frac{\varepsilon \operatorname{Sec}\varphi}{\operatorname{Tang}\varphi \operatorname{Sec}\varphi}. \qquad 8.\ \varepsilon\varphi = -\frac{\varepsilon \operatorname{Cosec}\varphi}{\operatorname{Cot}\varphi \operatorname{Cosec}\varphi}.$$

120. Corollarium 5.

Si $\varphi = \operatorname{Arc\,Sin} z$, vel $\varphi = \operatorname{Arc\,Cos} z$ denotet arcum circuli, cujus sinus vel cosinus sit z; erit $\operatorname{Sin}\varphi = z$ et $\operatorname{Cos}\varphi = \sqrt{(1-z^2)}$ casu primo, vel $\operatorname{Cos}\varphi = z$ et $\operatorname{Sin}\varphi = \sqrt{(1-z^2)}$ casu secundo: ob (119. §. 1. 2. n.) erit igitur

$$\varepsilon \operatorname{Arc\,Sin} z = \frac{\varepsilon z}{\sqrt{(1-z^2)}}. \quad \varepsilon \operatorname{Arc\,Cos} z = \frac{-\varepsilon z}{\sqrt{(1-z^2)}}.$$

121. Corollarium 6.

Si vero $\varphi = \operatorname{Arc\,Sin} v\, z$ vel $\varphi = \operatorname{Arc\,Cos} v\, z$ significet arcum, cujus sinus vel cosinus versus est z; erit $\operatorname{Sin} v\,\varphi = z$ et $\operatorname{Sin}\varphi = \sqrt{(2z-z^2)}$ casu primo, vel $\operatorname{Cos} v\,\varphi = z$ et $\operatorname{Cos}\varphi = \sqrt{(2z-z^2)}$ casu altero: per (119. §. 3. 4. n.) obtinebimus igitur

$$\varepsilon \operatorname{Arc\,Sin} v\, z = \frac{\varepsilon z}{\sqrt{(2z-z^2)}}. \quad \varepsilon \operatorname{Arc\,Cos} v\, z = \frac{-\varepsilon z}{\sqrt{(2z-z^2)}}.$$

122. Corollarium 7.

Si porro $\varphi = \operatorname{Arc\,Tang} z$ vel $\varphi = \operatorname{Arc\,Cot} z$ exprimat arcum, cujus tangens vel cotangens est z; erit $\operatorname{Tang}\varphi = z$ et $\operatorname{Sec}\varphi^2 = 1 + z^2$ casu primo,

mo, vel $\text{Cot}\,\varphi = z$ et $\text{Cosec}\,\varphi^2 = 1 + z^2$ casu secundo: debet itaque esse per (119. §. 5. 6. n.)

$$s\,\text{Arc Tang}\,z = \frac{sz}{1+z^2}, \qquad s\,\text{Arc Cot}\,z = \frac{-sz}{1+z^2}.$$

123. Corollarium 8.

Si denique $\varphi = \text{Arc Sec}\,z$ vel $\varphi = \text{Arc Cosec}\,z$ denotet arcum, cujus secans vel cosecans est z; erit $\text{Sec}\,\varphi = z$ et $\text{Tang}\,\varphi\,\text{Sec}\,\varphi = z\sqrt{(1-z^2)}$ casu primo, vel $\text{Cosec}\,\varphi = z$ et $\text{Cot}\,\varphi\,\text{Cosec}\,\varphi = z\sqrt{(1-z^2)}$: igitur per (119. §. 7. 8. n.) fiet.

$$s\,\text{Arc Sec.}\,z = \frac{sz}{z\sqrt{(1-z^2)}}. \qquad s\,\text{Arc Cosec}\,z = \frac{-sz}{z\sqrt{(1-z^2)}}.$$

124. Problema.

Dato exponente primae rationis differentialis certae functionis y, *invenire exponentes quarumvis altiorum rationum differentialium ejusdem functionis.*

Solutio.

Dato $sy = P$, quaeratur $s.sy = sP$ exponens primae rationis differentialis functionis datae P; erit $\overset{2}{s}y = sP$ exponens secundae rationis differentialis functionis y: invento $\overset{2}{s}y = sP = Q$, quaeratur $s.\overset{2}{s}y = sQ$ exponens primae rationis differentialis functionis Q; erit $\overset{3}{s}y = sQ$ exponens tertiae rationis differentialis functionis y: et generatim cognito exponente $\overset{n}{s}y = Z$ *ntae* rationis differentialis functionis y, quaeratur $s.\overset{n}{s}y = sZ$ exponens primae rationis differentialis functionis Z; erit $\overset{n+1}{s}y = sZ$ exponens proximae altioris $(n+1)$*esimae* rationis differentialis functionis y (93. §.).

125. Corollarium 1.

Cum nulla sit functio assignabilis unius variabilis absolutae, pro qua exponens primae rationis differentialis ope praecedentium principiorum nequeat determinari; sufficient eadem principia ad inveniendos exponentes omnium altiorum rationum differentialium quarumvis functionum unius variabilis absolutae (124. 94. §.).

126. Co-

126. Corollarium 2.

Dum functio y datur per ipsam variabilem absolutam x, pro qua fit $\varepsilon x = 1$; determinabuntur omnes exponentes differentiales εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, etc. per certas functiones ejusdem variabilis absolutae x, carentes exponente differentiali εx. Generatim vero recipiet functio y omnium ordinum rationes differentiales, si ad nullam rationem differentialem devenire licuerit, cujus exponens constans sit: utprimum autem exponens $\overset{n}{\varepsilon} y$ alicujus *ntae* rationis differentialis constanti cuipiam quantitati fuerit aequalis; erunt exponentes $\overset{n+1}{\varepsilon} y$, $\overset{n+2}{\varepsilon} y$, et sic porro altiorum rationum differentialium aequales nihilo (124. 80. §.). Sic e. gr. functio $y = \frac{1}{x}$ dabit $\varepsilon y = -\frac{1}{x^2}$; $\overset{2}{\varepsilon} y = \frac{2}{x^3}$; $\overset{3}{\varepsilon} y = -\frac{6}{x^4}$; et sic porro: functio $y = x^3$ vero dabit $\varepsilon y = 3x^2$; $\overset{2}{\varepsilon} y = 6x$; $\overset{3}{\varepsilon} y = 6$; $\overset{4}{\varepsilon} y = 0$, $\overset{5}{\varepsilon} y = 0$, et ita porro.

127. Corollarium 3.

Si pro functione $y = Az^a + Bz^b + Cz^c + \text{- - -} + Pz^p + \text{etc.}$ determines εy, $\overset{2}{\varepsilon} y$ per (124. 104. §.), eadem dein lege, quam coefficientes et exponentes in εy, $\overset{2}{\varepsilon} y$ sequuntur, exprimas exponentem $\overset{m}{\varepsilon} y$ *mtae* rationis differentialis functionis y, et hinc per (124. 104. §.) elicias $\overset{m+1}{\varepsilon} y$; invenies hunc quoque exponentem illi legi subjacere, unde idcirco per (31. §.) concludes, exponentem $\overset{r}{\varepsilon} y$ cujusvis *rtae* rationis differentialis, debere esse

$$\overset{r}{\varepsilon} y = \left\{ \begin{array}{l} a(a-1)(a-2) \text{ - - - } (a-r+1) A z^{a-r} \\ + b(b-1)(b-2) \text{ - - - } (b-r+1) B z^{b-r} \\ + c(c-1)(c-2) \text{ - - - } (c-r+1) C z^{c-r} \\ + \text{— — — — — — — — —} \\ \quad \text{— — — — — — — — —} \\ + p(p-1)(p-2) \text{ - - - } (p-r+1) P z^{p-r} \\ + \quad \text{etc.} \qquad\qquad \text{etc.} \end{array} \right.$$

128. Corollarium 4.

Ubi vero functio y differentianda data fuerit aut per unam variabilem z, aut per plures variabiles z, u, v, quae totidem functiones unius

varia-

variabilis absolutae x exprimant, debebunt exponentes differentiales variabilium z, u, v instar totidem functionum variabilis absolutae x tractari, quo fiet, ut non modo primi exponentes differentiales εz, εu, εv, sed etiam altiores, ut $\overset{2}{\varepsilon} z$, $\overset{3}{\varepsilon} z$, etc. in expressiones pro εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, etc. ingrediantur. Sic functio $y = z^2$ dabit $\varepsilon y = 2z\varepsilon z$ (104. §.); $\overset{2}{\varepsilon} y = 2z\overset{2}{\varepsilon} z + 2\varepsilon z^2$ (124. 105. §.); $\overset{3}{\varepsilon} y = 2z\overset{3}{\varepsilon} z + 6\varepsilon z\overset{2}{\varepsilon} z$ (124. 92. 105. 104. §.), et ita porro: functio $y = uv$ dabit $\varepsilon y = u\varepsilon v + v\varepsilon u$ (105. §.); $\overset{2}{\varepsilon} y = u\overset{2}{\varepsilon} v + v\overset{2}{\varepsilon} u + 2\varepsilon u\varepsilon v$ (124. 92. 105. §.); $\overset{3}{\varepsilon} y = u\overset{3}{\varepsilon} v + v\overset{3}{\varepsilon} u + 3\varepsilon u\overset{2}{\varepsilon} v + 3\varepsilon v\overset{2}{\varepsilon} u$ (124. 92. 105. 104. §.); et ita porro: functio autem $y = ux^2$, in qua est x variabilis absoluta, hinc $\varepsilon x = 1$, dabit $\varepsilon y = \varepsilon u x^2 + 2xu$ (105. 104. §); $\overset{2}{\varepsilon} y = 2u + 4x\varepsilon u + x^2\overset{2}{\varepsilon} u$ (124. 92. 105. 104. §.); $\overset{3}{\varepsilon} y = 6\varepsilon u + 6x\overset{2}{\varepsilon} u + x^2\overset{3}{\varepsilon} u$ (124. 92. 105. 104. §.), et sic porro.

129. Theorema.

Si, datis quantis Z, X *determinatae magnitudinis, et duobus aliis* U, V *variabilibus, quorum quodvis possit fieri minus quovis dato quanto, pro quibuslibet valoribus quantorum* U, V *fuerit* $Z > X \pm U$ *et simul* $Z < X \pm V$; *erunt quanta* Z, X *inter se aequalia.*

Demonstratio.

Impossibile est, ut sit $Z > X + U$ et simul $Z < X - V$ vel $Z < X + V$: si enim est $Z < X - V$; debet esse a fortiori $Z < X$, et $Z < X + U$: Si porro est $Z > X + U$; erit a fortiori $Z > X$; hinc, quia quanta Z, X sunt determinatae magnitudinis, extabit quantum tertium D magnitudinis pariter determinatae, pro quo fiat $Z = X + D$: cum autem, per hypothesin possit fieri $V < D$, poterit etiam fieri $X + V < Z$, eoque ipso est impossibile, ut, existente $Z > X + U$, sit simul $Z < X + V$ pro quibuslibet valoribus quantorum variabilium U, V.

Vi theorematis habebimus igitur binas sequentes conditiones possibiles, atque ex his evidenter potest demonstrari absoluta necessitas aequalitatis quantorum Z, X.

I. $Z > X - U$ et simul $Z < X + V$.
II. $Z > X - U$ et simul $Z < X - V$.

 Si

Si enim quanta Z, X non essent aequalia; deberet esse $Z > X$, vel $Z < X$, quorum utrumque pugnat cum assumtis conditionibus. Nam pro $Z > X$ absurda esset conditio secunda in (II), extaretque differentia $Z - X = D$ determinatae magnitudinis, pro qua fieret $Z = X + D$, hinc $D < V$ ob secundam conditionem in (I): igitur, cum per hypothesin possit fieri $V < D$, posset esse $D < V$ et simul $D > V$. Porro pro $Z < X$ esset $X = Z + D$ pro certo quanto D determinatae magnitudinis, hinc $Z = X - D$, et ideo $D < U$ ob primam conditionem in (I) et (II): quare, cum per hypothesin possit fieri $U < D$, posset esse $U < D$ et simul $U > D$.

130. Corollarium 1.

Ex hoc principio generali methodi exhaustionis antiquorum sequitur aliud principium analyticum, quod insignem habet usum in plurimis disquisitionibus maximi momenti: si nimirum $Z, X, P, Q, R, - - - p, q, r$, etc. sint quanta independentia a variabili ω, sitque pro quovis valore hujus variabilis $Z > X + p\omega + q\omega^2 + r\omega^3 +$ etc., et simul $Z < X + P\omega + Q\omega^2 + R\omega^3 +$ etc.; debebit esse $Z = X$ (76. 129. §.).

131. Corollarium 2.

Quamobrem, si differentia Δy certae functionis y relatae ad variabilem absolutam x ita dependeat a differentia Δx ejusdem variabilis, ut pro certis quantitatibus $E, F, G, - - - e, f, g$, etc. independentibus a Δx et quovis possibili valore differentiae Δx (15. §.) debeat esse $\Delta y > X\Delta x + e\Delta x^2 + f\Delta x^3 + g\Delta x^4 +$ etc. et simul $\Delta y < X\Delta x + E\Delta x^2 + F\Delta x^3 + G\Delta x^4 +$ etc., quia eo ipso debet etiam fieri $\varepsilon y > X + (e-k)\Delta x + (f-l)\Delta x^2 + (g-m)\Delta x^3 +$ etc., et $\varepsilon y < X + (E-k)\Delta x + (F-l)\Delta x^2 + (G-m)\Delta x^3 +$ etc. (84. §.); erit $\varepsilon y = X$ exponens differentialis functionis y (130. §.), cui $\varepsilon y = X\varepsilon x$ poterit substitui (81. §.).

132. Corollarium 3.

Huic principio (131. §.) innititur methodus quaerendi exponentes differentiales functionum incognitarum, qua in sequentibus frequenter utemur; quacunque vero methodo, hac aut alia simili, determinatus fuerit exponens differentialis εy, debebit is sumi cum signis contrariis, si constet, functionem y decrescere, crescente variabili (18. 79.).

133. De-

133. Definitio.

Quantitates variabiles u, v *dependent a se invicem*, si quoties alterutra ipsarum mutatur, ex hac mutatione certa mutatio in alteram redundat: quodsi autem quantitates u, v tales sint, ut quaevis ipsarum possit mutari, quin ideo etiam altera mutari debeat, erunt eae *independentes a se invicem*.

134. Corollarium 1.

Quaevis duarum a se invicem dependentium variabilium u, v potest spectari instar certae functionis alterius variabilis; quocirca erit alterutra ipsarum variabilis absoluta; vel ambae aequabuntur duabus functionibus unius variabilis absolutae x (133. 4. §.).

135. Corollarium 2.

Omnis functio y plurium a se invicem dependentium variabilium censenda est referri ad unicam variabilem absolutam (134. §.), cujus differentiatio idcirco praecedentibus praeceptis subjacet.

136. Corollarium 3.

Si vero quantitates variabiles u, v sint independentes a se invicem, non poterit una ipsarum aequari certae functioni alterius (133. §.): quamobrem debebit esse quaevis ipsarum variabilis absoluta (4. §.); aut alterutra erit variabilis absoluta, et altera aequalis certae functioni alicujus variabilis absolutae x; vel ambae aequabuntur quibusdam functionibus duarum variabilium absolutarum x, z.

137. Corollarium 4.

Omnis functio y plurium a se invicem independentium variabilium u, v, z, etc. censenda est referri ad plures variabiles absolutas, ita ut proprie nunquam liceat ejusmodi functionem y ad unicam aliquam variabilem absolutam p relatam spectare, a cujus mutationibus pendeant mutationes functionis y, singularumque variabilium u, v, z, etc.

138. Corollarium 5.

Quoties tamen variabiles u, v, z, etc. in data functione y a se invicem independentes fuerint; licebit illas mutationes seorsim considerare, quae ex certis mutationibus variabilium u, v, z, etc. seorsim sumtarum in functionem y debent redundare, ita e. gr., ut jam u, v, jam u, z, jam vero v, z pro constantibus quantitatibus possint sumi, adeoque y instar certae functio-

nis unius variabilis, z casu primo, v casu secundo, et u casu tertio queat spectari (133. §.).

139. Definitio.

Nomine *primae rationis differentialis* functionis y plurium a se invicem independentium variabilium u, v, z, etc. intelligemus rationem, cujus exponens εy aequatur summae exponentium primarum rationum differentialium, quas functio y pro singulis seorsim sumtis variabilibus, u, v, z, etc. recipit (138. §.).

140. Corollarium 1.

Exponens n*tae* rationis differentialis functionis y plurium variabilium u, v, z commodissime exprimetur, vel simpliciter signo $\overset{n}{\varepsilon} y$, vel, quo ipsae variabiles in y contentae exhibeantur, signo ${}^{uvz}\overset{n}{\varepsilon} y$; signa vero ${}^{u}\overset{n}{\varepsilon} y$, ${}^{v}\overset{n}{\varepsilon} y$, ${}^{z}\overset{n}{\varepsilon} y$ exprimunt exponentes n*tarum* rationum differentialium functioni y pro variabilibus u, v, z seorsim sumtis (138. §.) debitarum: erit igitur $\varepsilon y = {}^{uvz}\varepsilon y = {}^{u}\varepsilon y + {}^{v}\varepsilon y + {}^{z}\varepsilon y$ (139. §.).

141. Corollarium 2.

Data functione y plurium a se invicem independentium variabilium u, v, z, invenietur exponens $\varepsilon y = {}^{uvz}\varepsilon y$ illius primae rationis differentialis, si per praecedentia praecepta quaerantur exponentes primarum rationum differentialium, quas functio y pro singulis seorsim sumtis variabilibus u, v, z reciperet, iidem dein exponentes in unam summam addantur (140. 139. §.).

142. Corollarium 3.

Sicut autem ex data functione y plurium a se invicem independentium variabilium u, v, z per praecedentia praecepta elicitur exponens εy illius primae rationis differentialis (141. §.); sic poterit ex hoc ipso exponente elici exponens $\overset{2}{\varepsilon} y$ secundae, ex isto vero exponens $\overset{3}{\varepsilon} y$ tertiae, ex hoc porro exponens $\overset{4}{\varepsilon} y$ quartae, etc. rationis differentialis ejusdem functionis y (139. 124. §.)

143. Corollarium 4.

Omnis functio plurium variabilium a se invicem dependentium u, v, z, etc. exprimi potest formula generali $y = Au^a v^b z^c$ etc. $+ Bu^d v^e z^f$ etc. $+ Cu^g v^h z^i$ etc. $+$ etc. (63. §.), eritque (135. §.)

εy

$$\varepsilon y = (c A u^a v^b z^{c-1} + f B u^d v^e z^{f-1} + i C u^g v^h z^{i-1} + \text{etc.})\, \varepsilon z$$
$$+ (b A u^a z^c v^{b-1} + e B u^d z^f v^{e-1} + h C u^d z^i v^{h-1} + \text{etc.})\, \varepsilon v$$
$$+ (a A v^b z^c u^{a-1} + d B v^e z^f u^{d-1} + g C v^h z^i u^{g-1} + \text{etc.})\, \varepsilon u$$
$$+ \quad \text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Ob (139. §.) eſt ergo $\varepsilon y = {}^z\varepsilon y + {}^v\varepsilon y + {}^u\varepsilon y$.

144. Corollarium 5.

Quamobrem methodus differentiandi functiones plurium a ſe invicem independentium variabilium in (141. 142. §.) praeſcripta, extendi poteſt generatim ad omnes functiones plurium variabilium, ſeu hae dependeant a ſe invicem, ſeu non (143. §.).

145. Theorema.

Exponens primae rationis differentialis cujusvis functionis y *duarum variabilium* u, v *debet habere formam functionis* $M\,\varepsilon u + N\,\varepsilon v$, *in qua ſemper ſit* $\frac{{}^v\varepsilon M}{\varepsilon v} = \frac{{}^u\varepsilon N}{\varepsilon u}$.

Demonſtratio.

Generatim poteſt poni $y = A u^a v^b + B u^c v^d + C u^e v^f + \text{etc.}$ (63. §.), unde pro $M = a A v^b u^{a-1} + c B v^d u^{c-1} + e C v^f u^{e-1} + \text{etc.}$, et $N = b A u^a v^{b-1} + d B u^c v^{d-1} + f C u^e v^{f-1} + \text{etc.}$, obtinetur ${}^u\varepsilon y = M\,\varepsilon u$, ${}^v\varepsilon y = N\,\varepsilon v$: igitur (140. 144. §.) ${}^{uv}\varepsilon y = M\,\varepsilon u + N\,\varepsilon v$, et $\frac{{}^v\varepsilon M}{\varepsilon v} = \frac{{}^u\varepsilon N}{\varepsilon u}$.

146. Corollarium 1.

Ex ${}^u\varepsilon y = M\,\varepsilon u$, ${}^v\varepsilon y = N\,\varepsilon v$, obtinetur ${}^v\varepsilon . {}^u\varepsilon y = {}^v\varepsilon M\,\varepsilon u$, ${}^u\varepsilon . {}^v\varepsilon y = {}^u\varepsilon N\,\varepsilon v$; cum igitur debeat eſſe ${}^u\varepsilon N\,\varepsilon v = {}^v\varepsilon M\,\varepsilon u$ (145. §.), quaelibet functio y duarum variabilium u, v ita eſt comparata, ut pro ea debeat fieri ${}^v\varepsilon . {}^u\varepsilon y = {}^u\varepsilon . {}^v\varepsilon y$.

147. Corollarium 2.

Si functio quaecunque y duarum variabilium u, v bis continuo differentietur, ita ut primum capiatur exponens differentialis functionis y pro ſola variabili u, tum exponens differentialis exponentis ${}^u\varepsilon y$ pro ſola variabili v, vel primum differentietur y pro ſola variabili v, tum ipſe exponens ${}^v\varepsilon y$ differentietur pro ſola variabili u; eadem utroque caſu obtinebitur functio formae generalis $X\,\varepsilon u\,\varepsilon v$, denotante X functionem independentem ab exponentibus differentialibus εu, εv (146. §.).

148. Co-

148. Corollarium 3.

Ex his facilis est transitus ad inventionem legum, quibus tam exponentes altiorum rationum differentialium functionibus duarum variabilium debitarum, quam exponentes rationum differentialium debitarum functionibus trium, quatuor, etc. variabilium subjacent. Sic, quia exponens primae rationis differentialis functionis y duarum variabilium u, v debet habere formam functionis $\varepsilon y = M \varepsilon u + N \varepsilon v$; habebit exponens *differentio-differentialis*, seu exponens secundae rationis differentialis ejusdem functionis y formam functionis $\overset{2}{\varepsilon} y = {}^{u}\varepsilon(M \varepsilon u + N \varepsilon v) + {}^{v}\varepsilon(M \varepsilon u + N \varepsilon v) = {}^{u}\varepsilon M \varepsilon u + M \overset{2}{\varepsilon} u + {}^{u}\varepsilon N \varepsilon v + {}^{v}\varepsilon M \varepsilon u + {}^{v}\varepsilon N \varepsilon v + N \overset{2}{\varepsilon} v$ (141. 142. §.); consequenter, posito ${}^{u}\varepsilon M = Q \varepsilon u$, ${}^{v}\varepsilon N = R \varepsilon v$, et ${}^{u}\varepsilon N = p \varepsilon u$, ${}^{v}\varepsilon M = q \varepsilon v$, erit semper,

$$\overset{2}{\varepsilon} y = M \overset{2}{\varepsilon} u + N \overset{2}{\varepsilon} v + P \varepsilon u \varepsilon v + Q \varepsilon u^2 + R \varepsilon v^2,$$

pro $Q = \frac{{}^{u}\varepsilon M}{\varepsilon u}$, $R = \frac{{}^{v}\varepsilon N}{\varepsilon v}$, et ob (145. §.)

$$P = p + q = \frac{{}^{v}\varepsilon M}{\varepsilon v} + \frac{{}^{u}\varepsilon N}{\varepsilon u} = 2\left(\frac{{}^{v}\varepsilon M}{\varepsilon v}\right) = 2\left(\frac{{}^{u}\varepsilon N}{\varepsilon u}\right).$$

149. Corollarium 4.

Exponens vero primae rationis differentialis cujusvis functionis y trium variabilium p, q, r continebitur formula generali $\varepsilon y = {}^{p}\varepsilon y + {}^{q}\varepsilon y + {}^{r}\varepsilon y$ (140. 141. §.) $= P \varepsilon p + Q \varepsilon q + R \varepsilon r$, ita ut, quia quaevis variabilium p, q, r seorsim sumta in ejusmodi differentiatione instar constantis quantitatis potest spectari, debeat esse $\frac{{}^{q}\varepsilon P}{\varepsilon q} = \frac{{}^{p}\varepsilon Q}{\varepsilon p}$, $\frac{{}^{r}\varepsilon P}{\varepsilon r} = \frac{{}^{p}\varepsilon R}{\varepsilon p}$, $\frac{{}^{r}\varepsilon Q}{\varepsilon r} = \frac{{}^{q}\varepsilon R}{\varepsilon q}$ (145. §.).

150. Corollarium 5.

Deinde si functio y trium variabilium p, q, r continuo ter differentietur, quocunque ordine haec differentiatio perficiatur, puta primum differentiando ipsam functionem y pro sola variabili p, tum differentiando exponentem ${}^{p}\varepsilon y$ pro sola variabili q, ac demum differentiando exponentem ${}^{q}\varepsilon . {}^{p}\varepsilon y$ pro variabili r, vel alio quovis ordine; eadem semper prodibit functio formae generalis $Z \varepsilon p \varepsilon q \varepsilon r$, exprimente Z functionem independentem ab exponentibus differentialibus εp, εq, εr: id enim ex (149. §.), ut (147. §.) ex (145. §.) potest colligi.

151. Theo-

151. Theorema.

Si capiatur mtus exponens differentialis ${}^{u}\overset{m}{\varepsilon}y$, *et ntus exponens differentialis* ${}^{v}\overset{n}{\varepsilon}y$ *functionis* y *duarum variabilium* u, v, *prior pro sola variabili* u, *et posterior pro sola* v, *tum quaeratur ntus exponens differentialis* ${}^{v}\overset{n}{\varepsilon}.{}^{u}\overset{m}{\varepsilon}y$ *exponentis* ${}^{u}\overset{m}{\varepsilon}y$ *pro sola variabili* v, *et mtus* ${}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{n}{\varepsilon}y$ *exponentis* ${}^{v}\overset{n}{\varepsilon}y$ *pro sola variabili* u; *debebit esse* ${}^{v}\overset{n}{\varepsilon}.{}^{u}\overset{m}{\varepsilon}y = {}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{n}{\varepsilon}y$.

Demonstratio.

1. Ponamus pro certo indice r esse ${}^{v}\varepsilon.{}^{u}\overset{r}{\varepsilon}y = {}^{u}\overset{r}{\varepsilon}.{}^{v}\varepsilon y$; erit etiam ${}^{u}\varepsilon.{}^{v}\varepsilon.{}^{u}\overset{r}{\varepsilon}y = {}^{u}\varepsilon.{}^{u}\overset{r}{\varepsilon}.{}^{v}\varepsilon y$: est autem ${}^{u}\varepsilon.{}^{v}\varepsilon.{}^{u}\overset{r}{\varepsilon}y = {}^{v}\varepsilon.{}^{u}\varepsilon\,{}^{u}\overset{r}{\varepsilon}y$ (146. §.) $= {}^{v}\varepsilon.{}^{u}\overset{r+1}{\varepsilon}y$, et ${}^{u}\varepsilon.{}^{u}\overset{r}{\varepsilon}.{}^{v}\varepsilon y = {}^{u}\overset{r+1}{\varepsilon}.{}^{v}\varepsilon y$: igitur debet esse ${}^{v}\varepsilon.{}^{u}\overset{r+1}{\varepsilon}.y = {}^{u}\overset{r+1}{\varepsilon}.{}^{v}\varepsilon y$. Hinc et ex (146. §.) per (31. §.) sequitur, pro quovis possibili indice k debere esse ${}^{v}\varepsilon.{}^{u}\overset{k}{\varepsilon}y = {}^{u}\overset{k}{\varepsilon}.{}^{v}\varepsilon y$.

2. Porro sit pro indeterminatis indicibus r, m, ${}^{v}\overset{r}{\varepsilon}.{}^{u}\overset{m}{\varepsilon}y = {}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{r}{\varepsilon}y$; erit etiam ${}^{v}\overset{r+1}{\varepsilon}.{}^{u}\overset{m}{\varepsilon}y = {}^{v}\varepsilon.{}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{r}{\varepsilon}y = {}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{r+1}{\varepsilon}y$ ob (1): quare, cum pro $r = 1$ re ipsa sit ${}^{v}\varepsilon.{}^{u}\overset{m}{\varepsilon}y = {}^{u}\overset{m}{\varepsilon}.{}^{v}\varepsilon y$ ob (1), debet pro quovis possibili indice n ob (31. §.) esse ${}^{v}\overset{n}{\varepsilon}.{}^{u}\overset{m}{\varepsilon}y = {}^{u}\overset{m}{\varepsilon}.{}^{v}\overset{n}{\varepsilon}y$.

152. Corollarium 1.

Hinc (151. §.) per (140. 141. §.) obtinebimus sequentes expressiones pro exponentibus differentialibus εy, $\overset{2}{\varepsilon}y$, $\overset{3}{\varepsilon}y$, $\overset{4}{\varepsilon}y$ functionis indeterminatae y duarum variabilium u, v.

$$\varepsilon y = {}^{u}\varepsilon y + {}^{v}\varepsilon y.$$

$$\overset{2}{\varepsilon}y = {}^{u}\overset{2}{\varepsilon}y + 2\,{}^{u}\varepsilon.{}^{v}\varepsilon y + {}^{v}\overset{2}{\varepsilon}y.$$

$$\overset{3}{\varepsilon}y = {}^{u}\overset{3}{\varepsilon}y + 3\,{}^{u}\overset{2}{\varepsilon}.{}^{v}\varepsilon y + 3\,{}^{u}\varepsilon.{}^{v}\overset{2}{\varepsilon}y + {}^{v}\overset{3}{\varepsilon}y.$$

$$\overset{4}{\varepsilon}y = {}^{u}\overset{4}{\varepsilon}y + 4\,{}^{u}\overset{3}{\varepsilon}.{}^{v}\varepsilon y + 6\,{}^{u}\overset{2}{\varepsilon}.{}^{v}\overset{2}{\varepsilon}y + 4\,{}^{u}\varepsilon.{}^{v}\overset{3}{\varepsilon}y + {}^{v}\overset{4}{\varepsilon}y.$$

153. Corollarium 2.

Lex, quam sequuntur termini singularum expressionum in (152. §.), manifesta est: si sumas expressionem exponentis differentialis $\overset{r}{\varepsilon}y$ pro indeter-

determinato indice r eidem lege conformem, tum ex illa elicias $\overset{r+1}{\varepsilon} y = \varepsilon . \overset{r}{\varepsilon} y$ per (140. 142. §.), invenies, facta per (151. §.) reductione, etiam exponentem differentialem $\overset{r+1}{\varepsilon} y$ ei legi subjacere: hinc igitur per (31. §.) sequetur, quemvis *ntum* exponentem differentialem cujuslibet functionis y duarum variabilium u, v sequenti formula contineri:

$$\overset{n}{\varepsilon} y = {}^{u}\overset{n}{\varepsilon} y + \frac{n}{1} . {}^{u}\overset{n-1}{\varepsilon} . {}^{v}\varepsilon y + \frac{n(n-1)}{1 . 2} . {}^{u}\overset{n-2}{\varepsilon} . {}^{v}\overset{2}{\varepsilon} y$$
$$+ \frac{n(n-1)(n-2)}{1 . 2 . 3} . {}^{u}\overset{n-3}{\varepsilon} . {}^{v}\overset{3}{\varepsilon} y + - - - - - - - -$$
$$+ \frac{n(n-1) - - - (n-r+1)}{1 . 2 - - - - - - r} . {}^{u}\overset{n-r}{\varepsilon} . {}^{v}\overset{r}{\varepsilon} y + - - - - - + {}^{v}\overset{n}{\varepsilon} y.$$

154. Corollarium 3.

Ex hac generalissima formula, ob ${}^{u}\overset{n}{\varepsilon}(uv) = v\overset{n}{\varepsilon}u$, ${}^{v}\overset{n}{\varepsilon}(uv) = u\overset{n}{\varepsilon}v$, et generatim ${}^{u}\overset{n-r}{\varepsilon}({}^{v}\overset{r}{\varepsilon}.uv) = \overset{n-r}{\varepsilon}u\,\overset{r}{\varepsilon}v$, sequitur formula pro *nto* exponente differentiali producti $y = uv$ ex duabus variabilibus u, v, nimirum,

$$\overset{n}{\varepsilon}.uv = v\overset{n}{\varepsilon}u + \frac{n}{1} . \overset{n-1}{\varepsilon}u\,\varepsilon v + \frac{n(n-1)}{1 . 2} . \overset{n-2}{\varepsilon}u\,\overset{2}{\varepsilon}v + - - - - -$$
$$+ \frac{n(n-1) - - - (n-r+1)}{1 . 2 - - - - - r} . \overset{n-r}{\varepsilon}u\,\overset{r}{\varepsilon}v + - - - + u\overset{n}{\varepsilon}v.$$

155. Problema.

Invenire generalem expressionem valoris, quem functio y variabilis z obtineat, si haec variabilis augeatur data quantitate ω.

Solutio.

Variabilis z vel est absoluta, vel aequalis certae functioni alicujus variabilis absolutae x (4. §.): quidquid ea sit, spectetur ut absoluta, quo fiat $\varepsilon z = 1$, tum in hac hypothesi quaerantur exponentes differentiales εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, $\overset{4}{\varepsilon} y$ etc.: si enim Y valorem denotet, quem functio y, variabili z abeunte in $z + \omega$, debet obtinere, erit per (64. 127. §.)

$$Y = y + \frac{\varepsilon y}{1}\omega + \frac{\overset{2}{\varepsilon} y}{1 . 2}\omega^2 + \frac{\overset{3}{\varepsilon} y}{1 . 2 . 3}\omega^3 + - - - + \frac{\overset{r}{\varepsilon} y}{1 . 2 . 3 \, . \, . \, . \, r}\omega^r.$$

Seu, quod in idem recidit, ob $\varepsilon z = 1$

$$Y = y + \frac{\omega}{1} . \frac{\varepsilon y}{\varepsilon z} + \frac{\omega^2}{1 . 2} . \frac{\overset{2}{\varepsilon} y}{\varepsilon z^2} + \frac{\omega^3}{1 . 2 . 3} . \frac{\overset{3}{\varepsilon} y}{\varepsilon z^3} + - - - + \frac{\omega^r}{1 . 2 . 3 \, . \, . \, . \, r} . \frac{\overset{r}{\varepsilon} y}{\varepsilon z^s}.$$

156. Co-

156. Corollarium 1.

Si $-\omega$ loco ω sumatur in (155. §.), prodibit inde sequens generalis expressio valoris, quem functio y data per variabilem z obtineret, si haec variabilis minueretur quantitate ω.

$$Y = y - \frac{\varepsilon y}{1}\omega + \frac{\overset{2}{\varepsilon} y}{1.2}\omega^2 - \frac{\overset{3}{\varepsilon} y}{1.2.3}\omega^3 + \dots \pm \frac{\overset{r}{\varepsilon} y}{1.2.3\dots r}\omega^r.$$

Vel ob $\varepsilon z = 1$.

$$Y = y - \frac{\omega}{1}\cdot\frac{\varepsilon y}{\varepsilon z} + \frac{\omega^2}{1.2}\cdot\frac{\overset{2}{\varepsilon} y}{\varepsilon z^2} - \frac{\omega^3}{1.2.3}\cdot\frac{\overset{3}{\varepsilon} y}{\varepsilon z^3} + \dots \pm \frac{\omega^r}{1.2\dots r}\cdot\frac{\overset{r}{\varepsilon} y}{\varepsilon z^r}.$$

157. Corollarium 2.

Quamobrem, positis coefficientibus numericis $\frac{1}{1} = \overset{1}{\alpha}$, $\frac{1}{1.2} = \overset{2}{\alpha}$, $\frac{1}{1.2.3} = \overset{3}{\alpha}$, - - - $\frac{1}{1.2\dots r} = \overset{r}{\alpha}$, erit generatim, prout variabilis z augetur, vel minuitur quantitate ω (155. 156. §.)

$$Y = y \pm \overset{1}{\alpha}\overset{1}{\varepsilon} y\,\omega + \overset{2}{\alpha}\overset{2}{\varepsilon} y\,\omega^2 \pm \overset{3}{\alpha}\overset{3}{\varepsilon} y\,\omega^3 + \dots \pm \overset{r}{\alpha}\overset{r}{\varepsilon} y\,\omega^r.$$

Vel

$$Y = y \pm \overset{1}{\alpha}\frac{\varepsilon y}{\varepsilon z}\omega + \overset{2}{\alpha}\frac{\overset{2}{\varepsilon} y}{\varepsilon z^2}\omega^2 \pm \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon z^3}\omega^3 + \overset{4}{\alpha}\frac{\overset{4}{\varepsilon} y}{\varepsilon z^4}\omega^4 \pm \dots \pm \overset{r}{\alpha}\frac{\overset{r}{\varepsilon} y}{\varepsilon z^r}\omega^r.$$

158. Corollarium 3.

Si, data functione Z duarum variabilium u, v, variabilis u maneat constans, et v abeat in $v+\Delta v$; obtinebit Z sequentem valorem X per (155. §.), ubi, v considerando instar variabilis absolutae, est $\varepsilon v = 1$: si porro in X maneat v constans, et u abeat in $u+\Delta u$; induet X sequentem valorem Z^1 (155. §.), sumta variabili u pro absoluta, quo fiat $\varepsilon u = 1$: si igitur ex prima serie eliciantur series pro εX, $\overset{2}{\varepsilon} X$, $\overset{3}{\varepsilon} X$, etc. eae dein substituantur in secunda serie, tum reductio fiat per (151. §.); obtinebitur tertia series, valorem Z^1 generatim exprimens, quem quaevis functio Z variabilium u, v eo casu induet, si hae ambae simul in $u + \Delta u$, $v + \Delta v$ abeant.

$$X = Z + \frac{{}^{v}\varepsilon Z\Delta v}{1} + \frac{{}^{v}\overset{2}{\varepsilon} Z\Delta v^2}{1.2} + \frac{{}^{v}\overset{3}{\varepsilon} Z\Delta v^3}{1.2.3} + \dots + \frac{{}^{v}\overset{r}{\varepsilon} Z\Delta v^r}{1.2\dots r}.$$

 $Z^1 =$

$$Z^{\mathrm{I}} = X + \frac{{}^{u}sX\Delta u}{1} + \frac{{}^{u}\overset{2}{s}X\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}X\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}X\Delta u^{r}}{1.2\ldots r}.$$

$$\begin{aligned} Z^{\mathrm{I}} = Z & + \frac{{}^{v}sZ\Delta v}{1} + \frac{{}^{v}\overset{2}{s}Z\Delta v^{2}}{1.2} + \frac{{}^{v}\overset{3}{s}Z\Delta v^{3}}{1.2.3} + - - - + \frac{{}^{v}\overset{r}{s}Z\Delta v^{r}}{1.2\ldots r} \\ & + \frac{{}^{u}sZ\Delta u}{1} + \frac{{}^{u}\overset{2}{s}Z\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}Z\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}Z\Delta u^{r}}{1.2\ldots r} \\ & + {}^{v}s\left(\frac{{}^{u}sZ\Delta u}{1} + \frac{{}^{u}\overset{2}{s}Z\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}Z\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}Z\Delta u^{r}}{1.2\ldots r}\right)\frac{\Delta v}{1} \\ & + {}^{v}\overset{2}{s}\left(\frac{{}^{u}sZ\Delta u}{1} + \frac{{}^{u}\overset{2}{s}Z\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}Z\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}Z\Delta u^{r}}{1.2\ldots r}\right)\frac{\Delta v^{2}}{1.2} \\ & + {}^{v}\overset{3}{s}\left(\frac{{}^{u}sZ\Delta u}{1} + \frac{{}^{u}\overset{2}{s}Z\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}Z\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}Z\Delta u^{r}}{1.2\ldots r}\right)\frac{\Delta v^{3}}{1.2.3} \\ & + - - - - - - - - - - \\ & \quad - - - - - - - - - - \\ & \quad - - - - - - - - - - \\ & + {}^{v}\overset{r}{s}\left(\frac{{}^{u}sZ\Delta u}{1} + \frac{{}^{u}\overset{2}{s}Z\Delta u^{2}}{1.2} + \frac{{}^{u}\overset{3}{s}Z\Delta u^{3}}{1.2.3} + - - - + \frac{{}^{u}\overset{r}{s}Z\Delta u^{r}}{1.2\ldots r}\right)\frac{\Delta v^{r}}{1.2\ldots r}. \end{aligned}$$

159. Corollarium 4.

Pari prorſus ratione quaeri poteſt expreſſio generalis valoris, quem functio Z trium aut plurium variabilium u, v, z, etc., his ſimul in $u+\Delta u$, $v+\Delta v$, $z+\Delta z$, etc. abeuntibus, obtineret, pro tribus variabilibus determinabitur ea ex jam inventa expreſſione Z^{I} pro duabus variabilibus (158. §.), ſi in Z^{I} cogitetur $z+\Delta z$ loco z poni, expreſſio dein ex Z^{I} per (155. §.) derivata per (151. §.) congrue reducatur.

CAPUT III.

DE PALMARIIS QUIBUSDAM APPLICATIONIBUS CALCULI DIFFERENTIALIS AD THEORIAM FUNCTIONUM.

160. Definitio.

Omnis aequatio, translatis ejus terminis ad unum membrum, induet formam aequationis $Z=0$. Si Z fuerit functio unius aut plurium variabilium, erit $Z=0$ aequatio *determinata* casu primo, et *indeterminata* casu secundo. Utroque porro casu appellabitur aequatio $Z=0$ *algebraica* vel *transcendens*, prout fuerit Z functio pure algebraica, vel saltem ex parte transcendens (5. §.).

161. Problema.

Datam aequationem $VX^{\frac{m}{n}} - U = 0$ *continentem functionem irrationalem* $X^{\frac{m}{n}}$ *convertere in aliam, quae omni irrationalitate afficiente functionem* X *careat.*

Solutio.

Cum sit $X^{\frac{m}{n}} = \frac{U}{V}$, erit differentiando $X^{\frac{m}{n}} = \frac{nX\partial\left(\frac{U}{V}\right)}{m\partial X}$; igitur erit $\frac{U}{V} = \frac{nX\partial\left(\frac{U}{V}\right)}{m\partial X}$ aequatio quaesita.

162. Corollarium 1.

Hac ratione poterit data quaevis aequatio $Z=0$ ab omni irrationalitate afficiente quantitates variabiles liberari: quare, cum praeterea omnis aequatio facile a fractionibus, si quas contineat, liberetur, licebit omnem aequationem algebraicam $Z=0$ ita considerare, functio integra et rationalis unius aut plurium variabi

163. Corollarium 2.

Expressio generalis omnis aequationis algebraicae determinatae poterit esse haec; $Z = x^m + A x^{m-1} + B x^{m-2} + C x^{m-3} + \ldots + P x + Q = 0$, aut alia hujus formae (162. §.): dicetur ea vero esse *1mi*, *2di*, *3tii*, vel generatim *mti ordinis*, prout functio Z fuerit *1mi*, *2di*, *3tii*, etc. ordinis. (9. §.).

164. Corollarium 3.

Aequatio autem algebraica indeterminata $Z = 0$ inter plures variabiles u, v, z poterit induere formam similem praecedenti (163. §.), ita ut quilibet terminus vel aliquam potentiam unius tantum variabilis, vel productum ex certis potentiis plurium variabilium complectatur (160. 162. §.): pertinebit idcirco aequatio $Z = 0$ ad *ordinem 1mum*, vel *2dum*, aut *3tium*, vel generatim *mtum*, prout fuerit functio Z *1mi*, vel *2di*, aut *3tii*, vel *mti* ordinis (8. §.). Eapropter, si aequatio indeterminata $Z = 0$ fuerit *mti* ordinis, in nullo termino poterit summa exponentium quantitatum variabilium u, v, z in eodem termino contentarum indicem ordinis m excedere (8. §.).

165. Definitio.

Radix aequationis determinatae (163. §.) vocatur is determinatus valor variabilis x, quo loco x assumto functio $Z = x^m + A x^{m-1} + B x^{m-2}$ + etc. re ipsa fit aequalis nihilo. Sic e. gr. functio $Z = x^4 + x^3 - 5x^2 + x - 6$ fiet pro quovis quatuor valorum $x = 2$, $x = -3$, $x = \sqrt{-1}$, $x = -\sqrt{-1}$ aequalis nihilo: quivis horum valorum dat igitur unam radicem aequationis $Z = x^4 + x^3 - 5x^2 + x - 6 = 0$.

166. Corollarium 1.

Cum functio Z nequeat aequari nihilo, nisi variabilis x determinatum aliquem valorem habeat, necesse est, ut omnis aequatio $Z = 0$ saltem unam habeat radicem, quidquid ea sit, realis, vel imaginaria (165. §.).

167. Corollarium 2.

Omnis aequatio quadratica $x^2 + Ax + B = 0$ habet duas radices, $x = -\frac{1}{2}A + \frac{1}{2}\sqrt{(A^2 - 4B)}$, et $x = -\frac{1}{2}A - \frac{1}{2}\sqrt{(A^2 - 4B)}$, quarum utraque erit realis, pro $A^2 = 4B$, et $A^2 > 4B$, vel utraque imaginaria pro $A^2 < 4B$ (165. §.).

168. Definitio.

Factores datae functionis integrae et rationalis $Z=x^m+Ax^{m-1}+Bx^{m-2}+---+Px+Q$ vocantur functiones pariter integrae et rationales, quae inter se multiplicatae producunt functionem Z: functiones ejusmodi *1mi*, vel *2di*, aut *3tii*, etc. ordinis (9. §.) appellantur factores *simplices*, vel *duplices* seu *quadratici*, aut *triplices* seu *cubici*, et ita porro.

169. Corollarium 1.

Functio integra et rationalis Z *divisibilis* est per aliam functionem integram et rationalem F, si ea per hanc divisa generet quotum aequalem functioni integrae et rationali X: omnis igitur functio integra et rationalis Z erit per quemvis suum factorem F divisibilis; et quaevis functio integra et rationalis F, per quam altera Z sit divisibllis, erit unus factor hujus functionis (168. §.).

170. Corollarium 2.

Pro $F=x-\mu$ in (169. §.) erit $Z=X(x-\mu)$, hinc $Z=0$ pro $x=\mu$: quilibet ergo factor simplex $x-\mu$ functionis integrae et rationalis $Z=x^m+Ax^{m-1}+---+Px+Q$ dat unam radicem aequationis determinatae $Z=0$. (165. §.).

171. Corollarium 3.

Omnis functio quadratica $\gamma x^2+\beta x+\alpha$ habet duos factores simplices

$$x\sqrt{\gamma}+\frac{1}{2\sqrt{\gamma}}(\beta-\sqrt{(\beta^2-4\alpha\gamma)}) \text{ et } x\sqrt{\gamma}+\frac{1}{2\sqrt{\gamma}}(\beta+\sqrt{(\beta^2-4\alpha\gamma)}),$$

utrumque realem, si sit $\beta^2=4\alpha\gamma$ vel $\beta^2>4\alpha\gamma$, aut utrumque imaginarium pro $\beta^2<4\alpha\gamma$ (168. §.).

172. Corollarium 4.

Arcus circuli φ, cujus sinus vel cosinus excedat radium, est impossibilis: quaelibet ergo functio quadratica $\gamma x^2+\beta x+\alpha$ constans ex imaginariis factoribus simplicibus poterit exprimi formula $q^2x^2-2pqx\,\mathrm{Cos}\,\varphi+p^2$, ipsi vero ejus factores imaginarii erunt $p(\mathrm{Cos}\,\varphi+\mathrm{Sin}\,\varphi\sqrt{-1})-qx$ et $p(\mathrm{Cos}\,\varphi-\mathrm{Sin}\,\varphi\sqrt{-1})-qx$ (171. §.).

173. Theorema.

Functio integra et rationalis $Z=x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+Dx^{m-4}+---+Kx+L$ *ordinis mti tot semper, nec plures,*

neque pauciores, habebit factores simplices formae $x-\mu$, quot unitates sunt in exponente m ordinis, ad quem ea pertinet.

Demonstratio.

1. Si sumantur numero m quantitates indeterminatae a, b, c, d, - - - k, l; erit productum $(x-a)(x-b)(x-c) - - - (x-k)(x-l) = x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + - - - + Ux + V$ certa functio ordinis *mti*, in qua quantitates a, b, c, - - - k, l, lege per naturam multiplicationis determinata inter se connexae constituent coefficientes P, Q, R, - - - U, V: quodsi ergo supponamus hanc functionem aequari datae functioni Z, quo fiat $P=A$, $Q=B$, $R=C$, - - - $U=K$, $V=L$ (29. §.); nascentur numero m aequationes inter totidem indeterminatas quantitates a, b, c, - - - k, l et datos coefficientes A, B, C, - - - K, L, ex quibus porro poterunt derivari aequationes determinatae, quarum quaevis unicam duntaxat quantitatem incognitam a, b, c, - - - k, l complectatur: quare, cum per quamvis harum aequationum determinatarum unus saltem valor incognitae a, b, c, - - - k, l, quam illa continet, perfecte sit determinatus (166. §.); extabunt eo ipso pro singulis incognitis a, b, c, - - - k, l certi valores, quibus loco a, b, c, - - - k, l assumtis re ipsa fiat $P=A$, $Q=B$, $R=C$, - - - $U=K$, $V=L$, hinc etiam $Z=(x-a)(x-b)(x-c) - - - (x-k)(x-l)$.

2. Est porro impossibile, ut functio Z praeter hos factores alium aliquem, ab illis distinctum, factorem simplicem $x-\alpha$ habeat. Nam pro certa functione X deberet fieri $Z=X(x-\alpha)$ per (168. §.); adeoque etiam $(x-a)(x-b)(x-c) - - - (x-k)(x-l) = X(x-\alpha)$: illud igitur productum fieret $=0$ pro $x=\alpha$, quod est impossibile, quin eo ipso aliquis factorum $x-a$, $x-b$, $x-c$, - - - $x-k$, $x-l$ fiat aequalis nihilo, puta $x-a=0$, hinc $x=a=\alpha$, contra hypothesin, cum $x-\alpha$ debeat esse factor a singulis illis factoribus distinctus.

174. Corollarium 1.

Quaelibet radix $x=\alpha$ aequationis determinatae $Z=x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+ - - - +Kx+L=0$ dat unum factorem simplicem $x-\alpha$ functionis Z (165. 173. §. 2. n.).

175. Corollarium 2.

Omnis aequatio determinata $Z=0$ ordinis *mti* (174. §.) habet radices numero m, nec plures, neque pauciores (170. 173. 174. §.).

176. Co-

176. Corollarium 3.

Quaelibet aequatio $Z = x^m + Ax^{m-1} + Bx^{m-2} + - - - + Kx + L = 0$ ordinis *mti*, ſi ejus radices numero m ſint a, b, c, - - - k, l, poteſt ſpectari inſtar producti ex m aequationibus *radicalibus* $x-a=0$, $x-b=0$, $x-c=0$, - - - - $x-k=0$, $x-l=0$, ita ut ſit $Z=(x-a)(x-b)(x-c) - - - - (x-k)(x-l)=0$ (175. 174. §.).

177. Corollarium 4.

Duae radices a, b dabunt aequationem *2di* ordinis $(x-a)(x-b)=x^2-(a+b)+ab=0$. Tres radices a, b, c dabunt aequationem *3tii* ordinis $(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$. Et quatuor radices a, b, c, d dabunt aequationem *4ti* ordinis $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2-(abc+abd+acd+bcd)x+abcd=0$ (176. §.).

178. Corollarium 5.

Et ſi generatim numero r radices a, b, c, - - - u generent aequationem $Z=x^r+Ax^{r-1}+Bx^{r-2}+ - - - +Lx^{r-(2n-2)}+Mx^{r-(2n-1)}+Nx^{r-2n}+ - - - +Tx+U=0$; naſcetur ex illa aequatio ordinis $r+1$ radices a, b, c, - - - u, v habitura, nimirum $Z(x-v)=x^{r+1}+(A-v)x^r+ - - - - +(M-vL)x^{r-(2n-2)}+(N-vM)x^{r-(2n-1)}+ - - - - +(U-vT)x-vU=0$ (176. §.).

179. Corollarium 6.

Lex, qua coefficientes ex radicibus componuntur, conſtans eſt in aequationibus *2di*, *3tii*, *4ti* ordinis (177. §): et ſi illa aſſumatur pro coefficientibus aequationis ordinis r; ſubiicietur ei eo ipſo etiam aequatio ordinis $r+1$ (178. §.). Hac ratione per (31. §.) facile evincemus, coefficientem termini ſecundi cujuslibet aequationis *mti* ordinis aequari ſummae omnium radicum ejusdem aequationis ſumtae cum ſigno contrario; terminum vero ultimum aequalem eſſe producto ex omnibus radicibus ſumto cum ſigno proprio vel contrario, prout exponens m ordinis, ad quem aequatio pertinet, eſt par vel impar; quemvis vero intermedium coefficientem, puta coefficientem *nti* termini aequari ſummae productorum ex quibuslibet numero $n-1$ radicibus ſumtae cum ſigno proprio vel contrario, prout index n ejusdem termini eſt numerus impar vel par.

180. Co-

180. Corollarium 7.

Quamobrem, ut aequatio aliqua $x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Kx + L = 0$ termino secundo Ax^{m-1} careat, necessarium est, ut ea nec omnes positivas, neque omnes negativas, sed positivas et negativas radices habeat, summaque radicum positivarum exaequet summam negativarum (179. §.).

181. Theorema.

Si functio $Z = x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Px + Q$ *pro certo valore* α *variabilis* x *positivum, et pro alio valore* β *negativum valorem obtineat; debebit una saltem radix aequationis* $Z = 0$ *inter* α *et* β *jacere.*

Demonstratio.

Sit $\alpha > \beta$ (idem ratiocinium est ad $\alpha < \beta$ adplicabile); necesse est, ut inter α et β innumeri valores, singuli minores valore α et majores valore β, locum habeant, atque inter omnes ejusmodi valores duo a, b extent, pro quibus functio Z ex valore positivo transeat in negativum: inter hos porro valores $a < b$ variabilis x intercedet necessario aliqua quantumcunque parva differentia $b - a = \omega$; quia secus functio Z pro $x = a = b$ valorem positivum simul et negativum obtineret, quod est impossibile. Quapropter dabitur inter a et b tertius aliquis valor $x = k$ variabilis x, pro quo functio Z nec positivum, neque negativum valorem indueret, qui eam idcirco redderet aequalem nihilo; quo ipso demonstratur, inter α et β unam radicem k aequationis $Z = 0$ jacere (165. §.).

182. Corollarium 1.

Aequatio $Z = 0$ habebit saltem unam radicem realem, si bini reales valores variabilis x extiterint, pro quorum uno functio Z obtineat valorem positivum, et pro altero negativum (181. §.).

183. Corollarium 2.

Si inter duas radices reales a, b datae aequationis $Z = 0$ nulla alia realis radix ejusdem aequationis jaceat; debebit functio Z pro quovis valore variabilis x jacente inter radices a, b valorem positivum, vel pro quovis valorem negativum obtinere (181. §.).

184. Corollarium 3.

Inter valores α et β variabilis x, pro quorum uno functio Z (181. §.) induat valorem positivum, et pro altero negativum; aut unica jacebit

radix

radix aequationis $Z = 0$, aut, si plures radices inter ipsos jacuerint, numerus omnium erit impar (181. 183. §.).

185. Corollarium 4.

Si vero functio Z pro utroque valore reali α, β variabilis x induat valorem positivum, vel pro utroque negativum; aut nulla jacebit radix aequationis $Z = 0$ inter valores reales α, β, aut numerus radicum realium inter ipsos jacentium erit par (181. 183. §.).

186. Definitio.

Evidens est, functionem y relatam ad variabilem absolutam x, quae, variabili x continuo crescente, aut perpetuo crescat, vel continuo decrescat, nunquam obtenturam valorem aliquem maximum vel minimum, certo valori variabilis x respondentem. Si autem functio y ita sit conformata, ut ea pro certo valore $x = v$ valorem V obtineat, majorem vel minorem singulis valoribus, quos ipsa pro quibuscunque aliis valoribus variabilis x, tam majoribus quam minoribus valore v, indueret; erit V *Maximum vel Minimum absolutum* functionis y. Si denique, variabili absoluta x continuo crescente, functio y pro certo valore $x = v$ talem valorem V recipiat, qui sit major vel minor singulis proxime sequentibus et praecedentibus valoribus, quos eadem functio pro aliquantillum majoribus et minoribus valoribus variabilis x obtineret, licet fors V non sit valor major vel minor quovis possibili valore functionis y; appellabitur V quoddam *Maximum respectivum* casu primo, et *Minimum respectivum* casu secundo, atque de hujuscemodi maximis et minimis sermo erit in sequentibus.

187. Corollarium 1.

Si functio y data per variabilem x obtineat valorem V pro $x = v$, valorem vero P pro $x = v + e$, et valorem Q pro $x = v - e$, sitque pro certo quantumcunque decrescente valore quantitatis e tam $V > P$, quam $V > Q$, vel $V < P$ et $V < Q$; erit V aliquod maximum casu primo, et aliquod minimum functionis y casu altero (186. §.).

188. Corollarium 2.

Et vicissim, si valor V, quem functio y pro valore $x = v$ induit, sit aliquod maximum aut minimum ejusdem functionis; debebit extare certus valor quantitatis e, pro quo quantumcunque decrescente functio y,

 posito

posito $x = v + e$, tum $x = v - e$, obtinebit valores, quorum uterque erit minor casu primo, et uterque major valore V casu secundo (186. §.).

Scholion 1.

Maxima minimaque functionum positiva a negativis distinguenda sunt. Constat, quantitatibus negativis $-A$, $-a$ nonnumquam valores inaequales ab Analystis ita tribui, ut quantitas negativa $-a$ major esse censeatur quantitate negativa $-A$, si positiva a minor sit quantitate positiva A. Qui hac ratione de quantitatibus negativis sentiunt, methodum investigandi valores quantitatum variabilium, qui datas illarum functiones reddant maximas vel minimas, ad generales quasdam regulas solent revocare, nec est, cur utrumque casum, maximorum nimirum minimorumque positivorum et negativorum, perpetuo distinguere cogantur. Nos interea cum aliis maxima minimaque positiva et negativa seorsim considerabimus, per valorem inter plures valores negativos maximum vel minimum eum constanter intellecturi, qui inter eosdem valores, si omnes forent positivi, esset maximus vel minimus: sic e. gr. inter numeros negativos $-2, -3, -4, -5, -6, -7$, erit nobis -2 minimus, et -7 maximus numerus negativus.

Scholion 2.

Methodum universalem et compendiariam investigandi maxima et minima quarumvis functionum suppeditat calculus differentialis: priusquam ejus principia exponam, paucis attingam aliam methodum, quam pro simplicioribus casibus iis, qui in calculo differentiali haud sunt versati, offert methodus exhaustionis antiquorum. Supponatur nimirum functio y variabilis x re ipsa esse maxima vel minima pro x; tum sumatur quantitas e indeterminata, quae ultra omnes terminos possit decrescere, ponaturque $x + e$, deinde $x - e$ loco x in y, quo fiat, ut y casu primo aliquem valorem P, et casu altero aliquem Q obtineat: pro casu maximi erit jam $y > P$ et simul $y > Q$; pro casu minimi vero debebit poni $y < P$ et $y < Q$ (188. §.), unde ope congruae reductionis facile determinabuntur binae expressiones $Z > X + a e + b e^2 + c e^3 +$ etc., et $Z < X + \alpha e + \beta e^2 + \gamma e^3 +$ etc., quae ob quantitates Z, X, a, b, c, - - - α, β, γ, etc. independentes ab e dabunt aequationem $Z = X$ (130. §.) determinantem valorem variabilis x, pro quo functio y sit maxima vel minima.

Exem-

Exempla.

Pro valore x, qui functionem $y = ax - bx^2$ reddat maximam, debebit esse

$$ax - bx^2 > a(x+e) - b(x+e)^2;$$
$$ax - bx^2 > a(x-e) - b(x-e)^2:$$

igitur erit

$$a < 2bx + be;\quad a > 2bx - be.$$

$$a = 2bx \text{ (130. §.), hinc } x = \frac{a}{2b}.$$

Pro valore x, qui functionem $y = \frac{x-n}{x^2+m}$ reddat maximam, debebit poni

$$\frac{x-n}{x^2+m} > \frac{x+e-n}{(x+e)^2+m}, \text{ et } \frac{x-n}{x^2+m} > \frac{x-e-n}{(x-e)^2+m}.$$

Hinc autem obtinebimus

$$x^2 - 2nx > m + ne - xe, \text{ et } x^2 - 2nx < m + xe - ne:$$

igitur $x^2 - 2nx = m$ (130. §.), $x = n \pm \sqrt{(m+n^2)}$.

Pro valore x, qui functionem $y = \frac{A+Bx^2}{x}$ reddat minimam, oportebit ponere

$$\frac{A+Bx^2}{x} < \frac{A+B(x+e)^2}{x+e}, \text{ et } \frac{A+Bx^2}{x} < \frac{A+B(x-e)^2}{x-e}.$$

Quamobrem erit

$$Bx^2 > A - Bxe \text{ et } Bx^2 < A + Bxe;$$

$$Bx^2 = A \text{ (130. §.), hinc } x = \pm\sqrt{\frac{A}{B}}.$$

189. Theorema.

Si exponens εy *primae rationis differentialis alicujus functionis* y *datae per variabilem* x, *hac pro absoluta sumta, ut sit* $\varepsilon x = 1$, *pro certo valore* $x = v$ *determinatum aliquem valorem positivum vel negativum obtineat; functio* y *nec maxima fiet neque minima pro valore* $x = v$.

Demonstratio.

Fiat $y = \pm V$ pro $x = v$, et $\varepsilon y = P$, $\overset{2}{\varepsilon} y = Q$, $\overset{3}{\varepsilon} y = R$, $\overset{4}{\varepsilon} y = S$, et sic porro pariter pro $x = v$; debebit functio y pro $x = v + e$ obtinere sequentem valorem A, et pro $x = v - e$ valorem B (157. §.)

 A=

$$A = \pm V + \overset{1}{\alpha} P e + \overset{2}{\alpha} Q e^2 + \overset{3}{\alpha} R e^3 + \overset{4}{\alpha} S e^4 + \text{etc.}$$

$$B = \pm V - \overset{1}{\alpha} P e + \overset{2}{\alpha} Q e^2 - \overset{3}{\alpha} R e^3 + \overset{4}{\alpha} S e^4 - \text{etc.}$$

Sit jam valor P exponentis εy debitus valori $x = v$ positivus; erit etiam $\overset{1}{\alpha} P e$ valoris positivi, et $-\overset{1}{\alpha} P e$ valoris negativi; pro certo valore e erit ergo pars valoris A sequens post $\pm V$ valoris positivi, et pars valoris B sequens post $\pm V$ erit valoris negativi (77. §.); consequenter valor $\pm V$ nec major erit, neque minor utroque valore A, B seorsim sumto, eoque ipso nequit esse $\pm V$ aliquod maximum, vel minimum functionis y (186. §. et 1. Schol. 188. §phi).

Si autem sit P valoris negativi, proinde $\overset{1}{\alpha} P e$ negativi, et $-\overset{1}{\alpha} P e$ positivi valoris; erit pars valoris A sequens post $\pm V$ valoris negativi, et pars valoris B sequens post $\pm V$ erit valoris positivi (77. §.): valor $\pm V$ nec ergo est major, neque minor utroque valore A, B seorsim accepto, quorum alterutrum locum haberet, si $\pm V$ esset aliquod maximum, vel minimum functionis y (186. §. et 1. Schol. 188. §phi).

190. Theorema.

Si exponens differentialis $\overset{2n+1}{\varepsilon} y$ indicis imparis $2n+1$ pro certo valore v variabilis x determinatum aliquem positivum aut negativum valorem obtineat, singuli autem exponentes differentiales εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, - - - $\overset{2n}{\varepsilon} y$ inferiorum ordinum pro eodem valore $x = v$ aequentur nihilo; functio y nec maxima fiet neque minima pro $x = v$.

Demonstratio.

Si supponamus, pro $x = v$ fieri $y = \pm V$, $\overset{2n+1}{\varepsilon} y = P$, $\overset{2n+2}{\varepsilon} y = Q$, $\overset{2n+3}{\varepsilon} y = R$, $\overset{2n+4}{\varepsilon} y = S$, etc. cum per hypothesin sit $\varepsilon y = 0$, $\overset{2}{\varepsilon} y = 0$, $\overset{3}{\varepsilon} y = 0$, - - - $\overset{2n}{\varepsilon} y = 0$ pro $x = v$; debebit functio y pro $x = v + e$ obtinere sequentem valorem A, et valorem B pro $x = v - e$ (157. §.).

$$A = \pm V + (\alpha P e + \overset{2}{\alpha} Q e^2 + \overset{3}{\alpha} R e^3 + \text{etc.})\ \overset{2n}{\alpha} \,.\, \overset{2n}{e}.$$

$$B = \pm V + (-\alpha P e + \overset{2}{\alpha} Q e^2 - \overset{3}{\alpha} R e^3 + \text{etc.})\ \overset{2n}{\alpha} \,.\, \overset{2n}{e}.$$

Hinc autem evidenter sequitur, eodem prorsus ratiocinio, quo in (189. §.) sumus usi, posse ostendi, valorem $\pm V$ functionis y debitum valori

valori v variabilis x in assumta hypothesi nec maximum esse, neque minimum.

191. Theorema.

Quodsi autem pro certo valore v variabilis x singuli exponentes εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, --- $\overset{2n-1}{\varepsilon} y$ fiant aequales nihilo, exponens vero $\overset{2n}{\varepsilon} y$ pro $x = v$ determinatum aliquem valorem obtineat; erit valor functionis y debitus valori $x = v$ aliquod maximum positivum, vel minimum negativum, si valor exponentis $\overset{2n}{\varepsilon} y$ pro $x = v$ fuerit negativus: si vero hic valor fuerit positivus; erit valor functionis y debitus valori $x = v$ aliquod minimum positivum vel maximum negativum.

Demonstratio.

Pro $x = v$ fiat $y = \pm V$, et $\overset{2n}{\varepsilon} y = P$, $\overset{2n+1}{\varepsilon} y = Q$, $\overset{2n+2}{\varepsilon} y = R$, $\overset{2n+3}{\varepsilon} y = S$, et sic porro: cum per hypothesin sit $\varepsilon y = 0$, $\overset{2}{\varepsilon} y = 0$, $\overset{3}{\varepsilon} y = 0$, --- $\overset{2n-1}{\varepsilon} y = 0$; debebit functio y pro $x = v + e$ habere sequentem valorem A vel C, et valorem B vel D pro $x = v - e$ (157. §.).

$$A = + V + (\overset{2n}{\alpha} P e + \overset{2n+1}{\alpha} Q e^2 + \overset{2n+2}{\alpha} R e^3 + \overset{2n+3}{\alpha} S e^4 + \text{etc.})\, e^{2n-1}.$$
$$B = + V + (\overset{2n}{\alpha} P e - \overset{2n+1}{\alpha} Q e^2 + \overset{2n+2}{\alpha} R e^3 - \overset{2n+3}{\alpha} S e^4 + \text{etc.})\, e^{2n-1}.$$
$$C = - V + (\overset{2n}{\alpha} P e + \overset{2n+1}{\alpha} Q e^2 + \overset{2n+2}{\alpha} R e^3 + \overset{2n+3}{\alpha} S e^4 + \text{etc.})\, e^{2n-1}.$$
$$D = - V + (\overset{2n}{\alpha} P e - \overset{2n+1}{\alpha} Q e^2 + \overset{2n+2}{\alpha} R e^3 - \overset{2n+3}{\alpha} S e^4 + \text{etc.})\, e^{2n-1}.$$

Sit jam $P = \overset{2n}{\varepsilon} y$ valoris positivi; erit etiam $\overset{2n}{\alpha} P e$ quantitas positiva, proinde debebit esse etiam summa omnium terminorum post $\pm V$ in singulis expressionibus A, B, C, D sequentium valoris positivi (77. §.): igitur dabitur quantitas e, pro qua, utcunque ea decrescat, erit tam $A > V$ quam $B > V$, vel tam $C < -V$, quam $D < -V$, id est, valor functionis y debitus valori $x = v$ erit aliquod minimum positivum V, aut maximum negativum $-V$ (186. §. et 1. Schol. 188. §phi).

Porro sit $P = \overset{2n}{\varepsilon} y$ valoris negativi; erit quoque $\overset{2n}{\alpha} P e$ quantitas negativa, eoque ipso erit summa omnium terminorum post $\pm V$ in singulis expressionibus A, B, C, D sequentium valoris negativi (77. §.): hoc igitur casu extabit quantitas e, pro qua, utc[illegible]rescente, debebit esse tam

$A < V$

$A < V$ quam $B < V$, vel $C > -V$ et $D > -V$, proinde erit valor functionis y debitus valori $x = v$ aut aliquod maximum positivum V, aut minimum negativum $-V$ (186. §. et Schol. 1.)

192. Problema.

Determinare valores variabilis x, pro quibus data functio y fiat maxima vel minima.

Solutio.

Sumta variabili x pro absoluta, positoque $\partial x = 1$, capiatur ∂y, et, quia exponens ∂y pro illo valore variabilis x, qui functionem y reddit maximam vel minimam, nullum valorem finitum potest habere 189. §.), ponatur $\partial y = 0$, tum quaerantur radices aequationis $\partial y = 0$: si enim aequatio haec radices reales a, b, c, etc. habeat; erit fors functio y pro quavis radice $x = a$, $x = b$, $x = c$, etc. maxima, vel minima, quod sequenti ratione licebit explorare.

Sumtis successive exponentibus differentialibus ∂y, $\partial^2 y$, $\partial^3 y$, - - - - $\partial^{2r} y$, $\partial^{2r+1} y$, ponatur in quovis $x = a$: si pro radice $x = a$ aequationis $\partial y = 0$ reliqui exponentes successive aequentur nihilo; incertum erit, an functio y pro $x = a$ fiat maxima vel minima: si autem deveniatur ad certum exponentem $\partial^{2r+1} y$ indicis imparis $2r + 1$, qui pro $x = a$ determinatum quempiam valorem obtineat; constabit eo ipso, functionem y pro $x = a$ nec maximam fieri, neque minimam (190. §.): quodsi vero deventum fuerit ad exponentem $\partial^{2r} y$ indicis 2r paris, qui pro $x = a$ valorem aliquem determinatum induat, positivum aut negativum; dabit functio y pro $x = a$ quoddam minimum positivum vel maximum negativum casu primo, aut aliquod maximum positivum vel minimum negativum casu altero (191. §.).

193. Corollarium 1.

Valorem variabilis x, qui functionem y reddat maximam vel minimam, ex aequatione $\partial y = 0$ quaeri oportere ideo solum asseruimus, quia certo constat, exponentem ∂y pro illo valore nulli valori finito posse aequari: verum hinc haud necessario sequitur, debere fieri $\partial y = 0$, cum possit esse $\partial y = \infty = \frac{K}{0}$, denotante K quantitatem aliquam constantem. Quamobrem, si non subsistat aequatio $\partial y = 0$; tentetur aequatio $\frac{1}{\partial y} = 0$,

cujus

cujus radices reales debeant dare valores variabilis x functionem y reddentes maximam vel minimam.

194. Corollarium 2.

Ceterum elucet quoque ex (187. §.), quo modo poſſit explorari, utrum functio y pro certa radice reali $x = a$ aequationis $\partial y = 0$ vel $\frac{r}{\partial y} = 0$ fiat maxima vel minima.

195. Corollarium 3.

Perſpicuum eſt, hanc methodum inveſtigandi maxima et minima proprie ad illas tantum functiones pertinere, quae uniformes ſunt (13. §.), aut pro uniformibus poſſunt haberi, quales ſunt e. gr. radices poteſtatum imparium, utpote unicum valorem realem exhibentes, et radices poteſtatum parium, quae, dum reales ſunt, binos quidem valores exhibent, unum poſitivum, et alterum negativum, quorum tamen quivis ſeorſim poteſt ſpectari.

196. Definitio.

Si continua differentiatione functionis $y = x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \cdots + Px + Q$ ex aequatione $y = 0$ deriventur aequationes $\partial y = 0$, $\overset{2}{\partial} y = 0$, $\overset{3}{\partial} y = 0$, $\overset{4}{\partial} y = 0$, etc.; vocabitur $y = 0$ aequatio *principalis*; reliquae autem aequationes appellabuntur ejus *aequationes differentiales*, $\partial y = 0$ *prima*, $\overset{2}{\partial} y = 0$ *ſecunda*, $\overset{3}{\partial} y = 0$ *tertia*, et ſic porro.

197. Corollarium 1.

Pro aequatione principali $X = 1 + Az + Bz^2 + Cz^3 + \cdots + Qz^{r-1} + Rz^r + Sz^{r+1} + Tz^{r+2} + Uz^{r+3} + \cdots + fz^{m-2} + gz^{m-1} + hz^m = 0$ erit ejus $r$ta aequatio differentialis (196. 127. §.).

$$\overset{r}{\partial} X = \left\{\begin{array}{l}
r(r-1)(r-2)(r-3) \cdots 5.4.3.2.1.R \\
+(r+1)r(r-1)(r-2) \cdots 5.4.3.2.Sz \\
+(r+2)(r+1)r(r-1) \cdots 5.4.3.Tz^2 \\
+(r+3)(r+2)(r+1)r \cdots 5.4.Uz^3 \\
+ \text{— — — — — — —} \\
\text{— — — — — — —} \\
\text{— — — — — — —} \\
+(m-2)(m-3) \cdots (m-r-1)fz^{m-(r+2)} \\
+(m-1)(m-2) \cdots (m-r)gz^{m-(r+1)} \\
+m(m-1)(m-2) \cdots (m-r+1)hz^{m-r}
\end{array}\right\} = 0$$

198. Co-

198. Corollarium 2.

Omnis aequatio differentialis $\overset{r}{s}y=0$ est aequatio principalis respectu aequationis differentialis $s.\overset{r}{s}y=\overset{r+1}{s}y=0$ uno gradu superioris (196. §.)

199. Corollarium 3.

Si aequatio $y=0$ praeter radices $a, b, --- p$ habeat radices aequales $\alpha, \alpha, \alpha, --- \alpha$ numero n, quo pro $U=(x-a)(x-b) --- (x-p)$ fiat $Y=U(x-\alpha)^n=0$ (176. §); erit prima ejus aequatio differentialis $sy=(x-\alpha)^n sU+nU(x-\alpha)^{n-1}=0$ (4. 81. §.), seu $sy=((x-\alpha)sU+nU)(x-\alpha)^{n-1}=0$, atque haec habebit radices aequales $\alpha, \alpha, --- \alpha$ numero $n-1$ (170. §.).

200. Corollarium 4.

Ex aequatione principali $y=0$ habente radices aequales $\alpha, \alpha, \alpha --- \alpha$ numero n derivari possunt aequationes differentialis $sy=0$, $\overset{2}{s}y=0$, $\overset{3}{s}y=0$, $---\ \overset{n-1}{s}y=0$, quarum prima habeat radices aequales α, α, α etc. numero $n-1$, et quaevis sequens una pauciores quam proxime praecedens, ultima vero $\overset{n-1}{s}y=0$ unam tantum radicem $=\alpha$ (199. 198. §.).

201. Corollarium 5.

Quoties igitur functio $y=U(x-\alpha)^n$ habuerit factorem $(x-\alpha)^n$, existente n numero integro positivo; toties erunt omnes exponentes differentiales $sy, \overset{2}{s}y, \overset{3}{s}y, --- \overset{n-1}{s}y$ aequales nihilo pro $x=\alpha$, exponens vero $\overset{n}{s}y$ pro $x=\alpha$, cum α nulla sit radix aequationis $\overset{n}{s}y=0$, valorem aliquem determinatum induet (200. 165. §.): functio y pro $x=\alpha$ nec ergo maxima fiet neque minima, si n sit numerus impar; maxima vero ea erit vel minima, si n fuerit numerus par (190. 191. §.).

202. Theorema.

Inter quasvis binas radices reales et inaequales p, q, *aequationis* $y=0$ *quarum una sit proxime major altera, ita ut inter eas nulla alia radix jaceat, jacebit una radix aequationis differentialis* $sy=0$ *exhibens unum valorem variabilis* x, *pro quo functio* y *dat quoddam maximum, positivum vel negativum.*

Demon-

Demonstratio.

Sit $p > q$, poterunt inter p et q innumeri valores variabilis x cogitari, quorum quivis minor est radice p et major radice q, pro singulis vero his valoribus variabilis x obtineret functio y certos valores, vel omnes positivos, vel omnes negativos, cum inter p et q nulla radix aequationis $y = o$ jaceat (183. §.): quare cum fiat $y = o$ tam pro $x = p$ quam pro $x = q$ (165. §.), facile hinc colligitur, certo valori variabilis x jacenti inter p et q respondere quoddam maximum functionis y, positivum vel negativum (186. §. et 1. Schol.); eum vero valorem debebit exhibere una radix aequationis $\partial y = o$ (192. §.).

203. Corollarium 1.

Sint a, b duae radicum realium aequationis $y = o$, a omnium maxima, et b proxime minor, quo fiat, ut inter illas nulla alia radix jaceat. Sumto valore $x = v$ majore maxima radice a, pro quo functio $y = x^m + A x^{m-1} + B x^{m-2} + - - - + Px + Q$ induat valorem positivum (78. §. 2. Schol.), cum inter v et a nulla jaceat radix aequationis $y = o$; debebit functio y pro quovis valore x jacente inter v et a manere valoris positivi (181. §.); igitur, quia inter quemvis ejusmodi valorem x jacentem inter v et a, et quemvis valorem x jacentem inter a et b unica radix a locum habet; debebit functio y pro quovis valore x inter a et b fieri valoris negativi (184. 185. §.). Inter radicem realem maximam a et proxime minorem radicem b aequationis $y = o$ jacet igitur una radix aequationis differentialis $\partial y = o$ exhibens unum valorem x, pro quo functio y dat quoddam maximum negativum (202. §.).

204. Corollarium 2.

Sint $i > k > l$ tres radices reales aequationis $y = o$, quarum secunda sit proxime minor prima, et tertia proxime minor secunda, ita ut nec inter i et k, neque inter k et l alia radix locum habeat; jacebunt inter illas duae radices aequationis differentialis, una inter i et k, altera vero inter k et l, ita quidem, ut alternatim una illarum det maximum positivum, altera vero maximum negativum functionis y, prima illud, et secunda istud, vel vicissim (202. 184. 185. §.).

205. Corollarium 3.

Aequatio $y = x^m + A x^{m-1} + B x^{m-2} + - - - + Px + Q = o$ mti ordinis habet m radices; aequatio autem differentialis $\partial y = m x^{m-1}$

$+(m-1)Ax^{m-2}+---+P=0$ debet habere radices una pauciores, nimirum numero $m-1$ (175. §.): quare, cum inter binas quantitates reales non nisi reales quantitates possint jacere, si omnes radices aequationis $y=0$ reales fuerint, erunt etiam omnes radices aequationis $\partial y=0$ reales: si autem aequatio principalis $y=0$ quocunque numero $r < m$ radices reales habeat; habebit aequatio $\partial y=0$ minimum $r-1$ radices reales (202. §.).

206. Corollarium 4.

Pro radicibus realibus $\overset{1}{a}, \overset{2}{a}, \overset{3}{a}, \overset{4}{a}, ----\overset{n}{a}, \overset{n+1}{a}, \overset{n+2}{a}$, etc. aequationis $y=0$ habeat aequatio $\partial y=0$ radices reales $\overset{1}{\alpha}, \overset{2}{\alpha}, \overset{3}{\alpha}, \overset{4}{\alpha}, ----\overset{n}{\alpha}, \overset{n+1}{\alpha}$, etc. (205. §.): si hae et illae ita sint dispositae, ut utrobique prima sit omnium maxima, tum reliquae se invicem ordine magnitudinis sequantur; jacebit $\overset{1}{\alpha}$ inter $\overset{1}{a}$ et $\overset{2}{a}$, $\overset{2}{\alpha}$ inter $\overset{2}{a}$ et $\overset{3}{a}$, et generatim $\overset{n}{\alpha}$ inter $\overset{n}{a}$ et $\overset{n+1}{a}$; quaevis porro radix, prima $\overset{1}{\alpha}$, tertia $\overset{3}{\alpha}$, quinta $\overset{5}{\alpha}$, etc. aequationis $\partial y=0$ dabit, sumta loco x, aliquod maximum negativum, radix autem secunda $\overset{2}{\alpha}$, quarta $\overset{4}{\alpha}$, sexta $\overset{6}{\alpha}$, etc. dabit quoddam maximum positivum functionis y (202. 203. 204. §.).

207. Corollarium 5.

Quoties aequatio differentialis $\partial y=0$ habuerit radices imaginarias; toties eas habebit etiam aequatio principalis $y=0$ (205. §.).

208. Corollarium 6.

Ex radicibus tamen realibus aequationis differentialis $\partial y=0$ non licet concludere in radices reales aequationis principalis $y=0$: possibile est, ut illae omnes sint reales, licet hae sint omnes imaginariae. Quodsi tamen radices reales aequationis differentialis $\partial y=0$ ejusmodi fuerint, ut, iis, prout se invicem ordine magnitudinis sequuntur, loco x successive substitutis, functio y maxima positiva et negativa alternatim det, ut in (206 §.); jacebit necessario inter quasvis binas ejusmodi radices una radix aequationis $y=0$ (181. §.); proinde, si aequatio $\partial y=0$ numero n ejusmodi radices reales habuerit, habebit aequatio $y=0$ ad minimum numero $n-1$ radices reales. Interea notandum est, aut omnes radices reales aequales, quas aequatio differentialis $\partial y=0$ potest habere, aut omnes exemta una in forma-

formatione illius judicii esse reiiciendas, prout numerus illarum est par vel impar (201. §.).

209. Corollarium 7.

Ut quantitas positiva inter duas alias quantitates jaceat, oportet utramque hanc quantitatem esse positivam, vel unam positivam et alteram negativam: dum ergo omnes radices datae aequationis $y=0$ reales fuerint, et aequatio differentialis $s\,y=0$ aliquot radices positivas habuerit; debebit habere aequatio $y=0$ totidem vel una plures radices positivas (205. 206. §.).

210. Theorema.

Numerus omnium radicum realium, quas data quaecunque aequatio $y=x^m+A x^{m-1}+B x^{m-2}+---+ P x \pm Q=0$ *habuerit, debebit esse par vel impar, prout exponens* m *ordinis, ad quem ea aequatio pertineat, fuerit par vel impar.*

Demonstratio.

1. Sumatur quantitas $\pm v$ ita, ut v major sit singulis radicibus realibus aequationis $y=0$, radicibus negativis, si quae adsint, spectatis instar positivarum, sitque $v^m > A v^{m-1} + B v^{m-2} + ---- + P v \pm Q$ (78. §. 2. Schol.)

2. Inter $+v$ et 0 jacebunt omnes radices positivae, inter $-v$ et 0 vero omnes negativae.

3. Sit jam m exponens par adeoque $(\pm v)^m$ valoris positivi; habebit functio y tam pro $x=+v$, quam pro $x=-v$ valorem positivum ob (1): pro $x=0$ autem fiet $y=\pm Q$. Existente igitur termino ultimo Q positivo; erit tam numerus radicum positivarum inter $+v$ et 0, quam numerus negativarum inter $-v$ et 0 jacentium (2) seorsim par (185. §.): si autem terminus ultimus Q sit negativus; erit numerus illarum, et harum radicum seorsim impar (184. §.). Numerus omnium radicum, positivarum et negativarum simul, est igitur par utroque casu.

4. Sit m exponens impar, adeoque $(\pm v)^m$ valoris positivi pro $+v$, et negativi pro $-v$: hoc casu dabit ergo $x=+v$ valorem positivum, et $x=-v$ valorem negativum pro y, ob (1); $x=0$ autem dabit $y=\pm Q$. Pro termino ultimo positivo $+Q$ erit igitur numerus radicum positivarum inter $x=+v$ et $x=0$ jacentium (2) par; numerus vero negativarum jacentium inter $x=-v$ et $x=0$ (2) impar (185. 184. §.): et pro termino

ultimo negativo $-Q$ erit numerus radicum positivarum inter $x=+v$ et $x=0$ (2) impar; numerus vero negativarum inter $x=-v$ et $x=0$ (2) par (184. 185. §.). Utroque ergo casu erit numerus omnium radicum, positivarum et negativarum simul, impar.

211. Corollarium 1.

Terminus ultimus Q aequationis $y=x^m+Ax^{m-1}+\cdots+Px+Q=0$, cujus omnes radices sint reales, erit positivus vel negativus, prout numerus radicum positivarum fuerit par vel impar. (210. 179. §.).

212. Corollarium 2.

Omnis aequatio $y=0$ *mti* ordinis habet radices numero m (175. §.): si inter illas dentur radices reales numero n, erunt reliquae numero $m-n$ imaginariae. Quare, cum numerus $m-n$ semper sit par, si uterque numerus m, n est par, aut uterque impar, perspicuum est, numerum radicum imaginariarum, quas aequatio $y=0$ potest habere, semper esse parem (210. §.).

213. Corollarium 3.

Aequatio $y=0$ ordinis paris m (211. §.) potest habere omnes radices imaginarias: aequatio vero $y=0$ ordinis imparis m debet habere saltem unam radicem realem (212. §.).

214. Problema.

Determinare relationem inter coefficientem cujusvis termini datae aequationis $y=0$ et coefficientes duorum terminorum adjacentium, quorum unus illum proxime praecedat, et alter proxime sequatur, pro casu radicum imaginariarum ejusdem aequationis.

Solutio.

1. Data sit aequatio $y=x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+\cdots+Qx^{m-r+1}+Rx^{m-r}+Sx^{m-r-1}+Tx^{m-r-2}+Ux^{m-r-3}+\cdots+fx^2+gx+h=0$, quaeraturque relatio inter quemvis *rtum* coefficientem R et coefficientes Q, S pro casu radicum imaginariarum.

2. Si ponas $x=\frac{1}{z}$ in (1), obtinebimus aequationem $X=1+Az+Bz^2+Cz^3+\cdots+Qz^{r-1}+Rz^r+Sz^{r+1}+Tz^{r+2}+Uz^{r+3}+\cdots+fz^{m-2}+gz^{m-1}+hz^m=0$.

3. Hinc

3. Hinc (2) invenies per (197. §.) aequationes differentiales ${}^{r-1}_{\;s}X=0$, ${}^{r}_{s}X=0$, ${}^{r+1}_{\;s}X=0$; et si quaevis harum dividatur per primum ejus terminum, tum ponatur $m-r+1=n$, $m-r=n-1$, $m-r-1=n-2$, fiet.

$$ {}^{r-1}_{\;s}X=1+\alpha z+\beta z^2+\gamma z^3+\delta z^4+ - - - +\mu z^n=0. $$

$$ {}^{r}_{s}X=1+az+bz^2+cz^3+ - - - - +uz^{n-1}=0. $$

$$ {}^{r+1}_{\;s}X=1+\mathfrak{A}z+\mathfrak{B}z^2+ - - - - - - +\mathfrak{C}z^{n-2}=0. $$

Pro $\alpha=\frac{rR}{Q}$; $\beta=\frac{(r+1)rS}{2Q}$; $\gamma=\frac{(r+2)(r+1)rT}{3.2Q}$;

$\delta=\frac{(r+3)(r+2)(r+1)rU}{4.3.2Q}$; - - - - - - - -

$\mu=\frac{m(m-1)(m-2) - - - (m-r+2)h}{(r-1)(r-2) - - - 4.3.2.1.Q}$.

$a=\frac{(r+1)S}{R}$; $b=\frac{(r+2)(r+1)T}{2R}$;

$c=\frac{(r+3)(r+2)(r+1)U}{3.2R}$; - - - - - - -

$u=\frac{m(m-1)(m-2) - - - (m-r+1)h}{r(r-1)(r-2) - - - - 4.3.2.R}$.

$\mathfrak{A}=\frac{(r+2)T}{S}$; $\mathfrak{B}=\frac{(r+3)(r+2)U}{2S}$;

- - - - - $\mathfrak{C}=\frac{m(m-1) - - - - (m-r)h}{(r+1)r(r-1) - - - 4.3.2.S}$.

4. Aequationes hae (3) pro $z=\frac{1}{x}$ abibunt in tres sequentes aequationes.

$$ L=x^n+\alpha x^{n-1}+\beta x^{n-2}+\gamma x^{n-3}+\delta x^{n-4}+ - - - +\mu=0. $$
$$ M=x^{n-1}+ax^{n-2}+bx^{n-3}+cx^{n-4}+ - - - - - - - +u=0. $$
$$ N=x^{n-2}+\mathfrak{A}x^{n-3}+\mathfrak{B}x^{n-4}+ - - - - - - - - - - +\mathfrak{C}=0. $$

5. Ex his porro aequationibus (4) per (197. §.) facile eliciuntur sequentes aequationes differentiales.

$$\overset{n-2}{\varepsilon}L = x^2 + \frac{2\alpha}{n}x + \frac{2\beta}{n(n-1)} = 0.$$

$$\overset{n-3}{\varepsilon}M = x^2 + \frac{2a}{n-1}x + \frac{2b}{(n-1)(n-2)} = 0.$$

$$\overset{n-4}{\varepsilon}N = x^2 + \frac{2\mathfrak{A}}{n-2}x + \frac{2\mathfrak{B}}{(n-2)(n-3)} = 0.$$

6. Iam vero fieri poteſt, ut aequationes quadraticae (5) habeant radices imaginarias, nimirum ſi ſit per (167. §.),

$$\frac{\alpha^2}{n^2} < \frac{2\beta}{n(n-1)}; \qquad \text{I. } R^2 < \frac{(m-r+1)(r+1)}{(m-r)^r} \cdot SQ.$$

$$\frac{a^2}{(n-1)^2} < \frac{2b}{(n-1)(n-2)}; \text{ hinc ob (3) II. } S^2 < \frac{(m-r)(r+2)}{(m-r-1)(r+1)} \cdot RT.$$

$$\frac{\mathfrak{A}^2}{(n-2)^2} < \frac{2\mathfrak{B}}{(n-2)(n-3)}; \qquad \text{III. } T^2 < \frac{(m-r-1)(r+3)}{(m-r-2)(r+2)} \cdot SU.$$

7. Relatio (I) indicabit ergo, utramque radicem aequationis $\overset{n-2}{\varepsilon}L = 0$ in (5) eſſe imaginariam; adeoque debebit hoc caſu etiam aequatio $L = 0$ in (4) habere ſaltem unum par radicum imaginariarum (207. 198. §.): cum igitur quaevis radix $x = \frac{1}{z}$ aequationis $L = 0$ in (4) det unam radicem $z = \frac{1}{x}$ aequationis $\overset{r-1}{\varepsilon}X = 0$ in (3), haec autem nequeat habere radices imaginarias, quin illas eo ipſo etiam aequatio $X = 0$ in (2) per (207. 198. §.) habeat; erunt in eadem hypotheſi etiam aliquae radices z aequationis $X = 0$ in (2), proinde etiam aliquae radices x aequationis $y = 0$ in (1) imaginariae.

8. Si dicas, nullum *rtum* coefficientem R aequationis $y = 0$ in (1) talem eſſe, ut is conditionem (I) in (6) impleat, ſed pro quovis ejusmodi coefficiente fore aut

$$R^2 = \frac{(m-r+1)(r+1)}{(m-r)^r} \cdot SQ, \text{ aut } R^2 > \frac{(m-r+1)(r+1)}{(m-r)^r} \cdot SQ,$$

proinde etiam in (6)

$$\frac{\alpha^2}{n^2} = \frac{2\beta}{n(n-1)}, \qquad \text{aut } \frac{\alpha^2}{n^2} > \frac{2\beta}{n(n-1)};$$

erunt quidem ambae radices aequationis $\overset{n-2}{\varepsilon}L = 0$ in (5) reales (167. §.): verum inde nondum ſequitur, aequationem $L = 0$ in (4) nullas habere

radices

radices imaginarias (208. 198. §.): si vero eo non obstante aequatio $L=0$ in (4) radices imaginarias habeat; habebit illas etiam aequatio $y=0$ in (1), ob rationes in (7) relatas.

9. Sicut (I) dat saltem unum par radicum imaginariarum aequationis $L=0$ in (4); sic quoque conditio (II.), cum ea indicet radices imaginarias aequationis $\overset{n-3}{s}M=0$ in (5), dabit saltem unum par radicum imaginariarum aequationis $M=0$ in (4) per (207. 198. §.): eapropter, quia quaevis radix $x=\frac{1}{z}$ aequationis $L=0$ vel $M=0$ in (4) det unam radicem $z=\frac{1}{x}$ aequationis $\overset{r-1}{s}X=0$ vel $\overset{r}{s}X=0$ in (3); dabit conditio (I) unum saltem par radicum imaginariarum aequationis $\overset{r-1}{s}X=0$, et conditio (II) indicabit saltem unum par radicum imaginariarum aequationis $\overset{r}{s}X=0$ in (3). Debet autem aequatio $\overset{r-1}{s}X=0$ eo ipso jam habere unum par radicum imaginariarum, quia id habet aequatio $\overset{r}{s}X=s.\overset{r-1}{s}X=0$ (207. §.): igitur habebit aequatio $\overset{r-1}{s}X=0$ ad minimum duas radices imaginarias ob conditionem (I), et duas ob conditionem (II). Verum hinc haud elucet, radices imaginarias aequationis $\overset{r-1}{s}X=0$ in (3), quas indicat conditio (I), ab illis ejus radicibus imaginariis, quas denotat conditio (II), distinctas esse; aliunde vero certo constat, eas non posse esse distinctas: secus enim posset habere aequatio $\overset{r-1}{s}X=0$ in (3) pro quocunque ejus exponente n, adeoque etiam pro $n=3$ quatuor radices imaginarias, quod est impossibile. Itaque binae conditiones (I) (II) in (6) tantundem indicant, quantum una illarum indicat, nimirum unum par radicum imaginariarum aequationis $y=0$ in (1), ob (7).

10. Quodsi autem conditiones (I) (III) in (6) locum habeant, quo casu radices aequationum $\overset{n-2}{s}L=0$, $\overset{n-4}{s}N=0$ in (5) debent esse imaginariae; habebunt etiam aequationes $L=0$, $N=0$ in (4) radices imaginarias (207. 198. §.): cum autem quaevis radix $x=\frac{1}{z}$ aequationis $L=0$ vel $N=0$ in (4) det unam radicem $z=\frac{1}{x}$ aequationis $\overset{r-1}{s}X=0$ vel $\overset{r+1}{s}X=0$ in (3); habebit quoque aequatio $\overset{r-1}{s}X=0$ saltem unum par radicum

radicum imaginariarum ob conditionem (I), et aequatio $\overset{r+1}{s}X = 0$ pariter unum par ob conditionem (III). Cum porro aequatio $\overset{r+1}{s}X = 0$ ſit 2*da* aequatio differentialis $\overset{2}{s}\,.\,\overset{r-1}{s}X = 0$ aequationis $\overset{r-1}{s}X = 0$ (196. §.); debet aequatio $\overset{r-1}{s}X = 0$ ea ipſo ſolum unum par radicum imaginariarum habere, quia id habet aequatio $\overset{r+1}{s}X = 0$ (207. 198. §.): aequatio igitur $\overset{r-1}{s}X = 0$ habet unum par radicum imaginariarum ob conditionem (I), et aliud par ob conditionem (III), atque iſtud par ab illo diſtinctum eſt. Nam ante omnia evidens eſt, ut conditio (III) locum habeat, debere eſſe $n > 3$; adeoque aequatio $\overset{r-1}{s}X = 0$ in (3) debet eſſe minimum 4*ti* gradus: hinc autem elucet, ipſam capacem eſſe quatuor radicum imaginariarum. Deinde certum eſt, coefficientes cujuslibet aequationis ab ejus radicibus certa lege dependere (179. §.), coefficientesque α, $\mathfrak{A}$ ob (3) per diverſos coefficientes aequationis principalis $y = 0$ determinari: igitur debent etiam radices imaginariae aequationis $\overset{r-1}{s}X = 0$ in (3), quarum indicium eſt conditio (I), diverſae eſſe ab ejus radicibus imaginariis, quas conditio (III) in (6) indicat. Ut primum vero aequatio $\overset{r-1}{s}X = 0$ in (3) ob conditiones (I) (III) duo habet paria radicum imaginariarum; habet etiam aequatio $X = 0$ in (2), proinde etiam $y = 0$ in (1), ſaltem duo paria radicum imaginariarum (207. 198. §.).

215. Corollarium.

His principiis nititur regula Neutoniana ad explorandas radices imaginarias datae cujuscunque aequationis inventa. Singulis nimirum terminis datae aequationis, exceptis termino primo et ultimo, ſuperſcribantur fractiones, prout eos ſequens exemplum exhibet: exploretur deinde, an quadratum cujusvis coefficientis minus ſit fractione ſuperſcripta ducta in productum coefficientium adjacentium, vel non: caſu primo ſubſcribatur eidem termino ſignum —, et caſu altero ſignum +, terminis vero primo et ultimo ſemper ſubſcribatur ſignum +. Numerus enim alternationum ſignorum ſubſcriptorum indicabit, aequationem datam ad minimum totidem radices imaginarias habere.

$$x^m + \overset{\frac{2m}{1(m-1)}}{Ax^{m-1}} + \overset{\frac{3(m-1)}{2(m-2)}}{Bx^{m-2}} + \overset{\frac{4(m-2)}{3(m-3)}}{Cx^{m-3}} + \overset{\frac{5(m-3)}{4(m-4)}}{Dx^{m-4}} + \ldots = 0.$$

216. Theo-

216. Theorema.

In quavis aequatione 2di gradus, cujus ambae radices reales sunt, tot semper adsunt signorum +— vel —+ alternationes, quot radices positivae, et tot successiones ++ vel ——, quot radices negativae.

Demonstratio.

Sequentes quatuor formulae exhibent omnes casus possibiles.

I. $x^2+Ax+B=0$. II. $x^2-Ax-B=0$.
III. $x^2-Ax+B=0$. IV. $x^2+Ax-B=0$.

Iam vero in (II) et (IV) necessario debet esse una radix positiva et altera negativa (179. §.), sicut una adest alternatio et una successio signorum. In (III) debet esse utraque radix positiva (179. §.), sicut duae adsunt signorum alternationes. In (I) demum sunt ambae radices negativae (179. §.), sicut binae signorum successiones.

217. Theorema.

Quoties omnes radices datae aequationis $y = 0$ *fuerint reales, ejusque rta aequatio differentialis* $\overset{r}{s}y=0$ *tot signorum alternationes quot radices positivas habuerit; toties debebit quoque proxime inferior aequatio differentialis* $\overset{r-1}{s}y = 0$ *totidem signorum alternationes habere, quot ea habet radices positivas*

Demonstratio.

1. Aequatio differentialis (r—1)*esima* fit $\overset{r-1}{s}y = x^n \pm ax^{n-1} \pm bx^{n-2} \pm cx^{n-3} \pm dx^{n-4} + \ldots \pm px \pm q = 0$.

2. Erit *rta* aequatio differentialis $\overset{r}{s}y = nx^{n-1} \pm (n-1)ax^{n-2} \pm (n-2)bx^{n-3} \pm (n-3)cx^{n-4} + \ldots \pm p = 0$.

3. Omnes signorum alternationes et successiones, quas aequatio (2) complectitur, debent quoque in aequatione (1) contineri.

4. Aequatio (2) habeat alternationes signorum, proinde etiam radices positivas numero m, qui par sit, aut impar.

5. Si m in (4) est numerus par; terminus p in (2) debet esse positivus, adeoque $+p$ (211. §.): prout igitur adest $+q$ vel $-q$ in (1), habet aequatio (1) aut totidem, aut una plures signorum alternationes, quam aequatio (2), nimirum numero m, aut $m+1$, ob (3).

6. Habet vero aequatio (1) ad minimum totidem radices reales positivas, quot aequatio (2) ob (198. 209. §.); consequenter habet (1) ad minimum numero m radices positivas ob (4): cum igitur m sit numerus par; debet (1) habere $m+1$ vel m radices positivas, prout $-q$ vel $+q$ adest in (1) per (211. §.). Aequatio (1) tot igitur habet positivas radices, quot signorum alternationes (5).

7. Si vero numerus m in (4) alternationum signorum, et radicum positivarum aequationis (2) impar est; necesse est, ut ejus terminus ultimus sit $-p$ (211. §.): prout est ergo $+q$ vel $-q$ in (1), habet aequatio (1) alternationes signorum numero $m+1$ vel m, ob (3).

8. Deinde habet aequatio (1) ad minimum m radices positivas, sicut aequatio (2) ob (5) per (198. 209. §.): quare, cum m sit impar numerus, debet aequatio (1) habere radices positivas numero $m+1$ vel m, prout ea $+q$ vel $-q$ habet pro termino ultimo (211. §.); adeoque tot debet ea habere radices positivas, quot signorum alternationes, ob (7).

218. Corollarium 1.

Si aequatio $y=0$ ordinis *mti* omnes radices reales habeat; habebit etiam quaevis aequationum differentialium $\varepsilon y=0$, $\overset{2}{\varepsilon} y=0$, $\overset{3}{\varepsilon} y=0$, - - - - - $\overset{m-1}{\varepsilon} y=0$ omnes radices reales (205. 198. §.), cumque ultima aequatio $\overset{m-1}{\varepsilon}\varepsilon y=0$ debeat esse quadratica, habebit ea tot signorum alternationes, quot radices positivas (216. §.): igitur habebit etiam quaevis praecedentium aequationum, proinde ipsa quoque aequatio principalis $y=0$ tot alternationes signorum, quot radices positivas (217. §.).

219. Corollarium 2.

Aequatio ordinis *mti* habet radices numero m, et terminos numero $m+1$, quorum quivis bini contigui dant unam signorum alternationem vel unam successionem, quocirca debet esse m numerus omnium alternationum et successionum simul; quare, si aequatio *mti* ordinis numero n radices positivas totidemque alternationes signorum habeat; habebit ea $m-n$ radices negativas, et totidem signorum successiones, si quidem omnes radices sint reales (218. §.).

Scho-

Scholion.

Cum quaelibet radix μ aequationis $y = 0$ debeat dare unum factorem simplicem $x - \mu$ functionis y (174. §.); quidquid hactenus de radicibus generatim demonstratum est, ad illustrandam etiam naturam factorum simplicium, in quos functiones integrae et rationales sunt resolubiles (173. §.), pertinet. Est autem resolutio functionum in factores simplices maximi momenti in calculo integrali; quocirca optandum foret, ut extaret aliqua methodus eam omni casu perficiendi: interea certum est, admissa resolutione omnis aequationis determinatae $y = 0$, quae inventione omnium illius radicum, reales sint, aut imaginariae, perficitur; eo ipso concedi etiam resolutionem cujuslibet functionis integrae et rationalis y in suos factores simplices (176. §.). Artificia autem, quae ad detegendas radices datarum aequationum hactenus excogitare licuit, admodum imperfecta sunt; qui ea noscere cupiant, scripta adeant, quorum praecipua in introductione commemoravi.

220. Theorema.

Si functio fracta $\frac{P}{Q}$ data per variabilem z ejus sit indolis, ut illa pro determinato quopiam valore $z = a$, evanescente numeratore P et denominatore Q, abeat in $\frac{0}{0}$, necesse est, ut ipsa pro $z = a$ eundem valorem habeat, quem quotus $\frac{\varepsilon P}{\varepsilon Q}$ pro $z = a$ obtineret.

Demonstratio.

1. Si pro $z = a$ exponentes differentiales $\varepsilon P, \overset{2}{\varepsilon} P, \text{----} \varepsilon Q, \overset{2}{\varepsilon} Q$, etc. obtineant valores $\varepsilon P^{\mathrm{I}}, \overset{2}{\varepsilon} P^{\mathrm{I}}, \text{----} \varepsilon Q^{\mathrm{I}}, \overset{2}{\varepsilon} Q^{\mathrm{I}}$, etc.; obtinebit functio $\frac{P}{Q}$ pro $z = a + \omega$ per (157. §.) sequentem valorem.

$$\frac{\overset{1}{\alpha}\varepsilon P^{\mathrm{I}} + \overset{2}{\alpha}\omega \overset{2}{\varepsilon} P^{\mathrm{I}} + \text{---} + \overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon} P^{\mathrm{I}}}{\overset{1}{\alpha}\varepsilon Q^{\mathrm{I}} + \overset{2}{\alpha}\omega \overset{2}{\varepsilon} Q^{\mathrm{I}} + \text{---} + \overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon} Q^{\mathrm{I}}}.$$

2. In eadem hypothesi debebit functio $\frac{P}{Q}$ pro $z = a - \omega$ sequentem valorem habere (157. §.).

$$\frac{-\overset{1}{\alpha}\varepsilon P + \overset{2}{\alpha}\omega \overset{2}{\varepsilon} P^{\mathrm{I}} - \text{---} \pm \overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon} P^{\mathrm{I}}}{-\overset{1}{\alpha}\varepsilon Q^{\mathrm{I}} + \overset{2}{\alpha}\omega \overset{2}{\varepsilon} Q^{\mathrm{I}} - \text{---} \pm \overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon} Q^{\mathrm{I}}}.$$

 3. Iam

3. Iam vero vel functio $\frac{P}{Q}$ crescit, vel decrescit, crescente variabili z: casu primo erit ejus valor debitus valori $z = a$ minor valore (1), et major valore (2); casu autem altero erit is major valore (1), et minor valore (2) pro certa quantumcunque parva quantitate ω: quodsi igitur V denotet valorem functionis $\frac{P}{Q}$ debitum valori $z = a$; dabuntur certi coefficientes $K, L, M, \ldots k, l, m$, etc., pro quibus ob (1) (2) erit utroque casu

$$V > \frac{\varepsilon P^{\mathrm{I}}}{\varepsilon Q^{\mathrm{I}}} + k\omega + l\omega^2 + m\omega^3 + \text{etc.}; \text{ et } V < \frac{\varepsilon P^{\mathrm{I}}}{\varepsilon Q^{\mathrm{I}}} + K\omega + L\omega^2 + M\omega^3 + \text{etc.}$$

Eapropter erit necessario $V = \frac{\varepsilon P^{\mathrm{I}}}{\varepsilon Q^{\mathrm{I}}}$ (130. §.).

221. Corollarium 1.

Si pro $z = a$ etiam quivis $r$to inferior exponens differentialis tam functionis P, quam functionis Q fiat aequalis nihilo; debebunt in (n. 1. 2.) omnes termini praecedentes terminos, in quibus continentur exponentes differentiales $\overset{r}{\varepsilon} P^{\mathrm{I}}$, $\overset{r}{\varepsilon} Q^{\mathrm{I}}$ negligi; valores idcirco functionis $\frac{P}{Q}$ debiti valoribus $a + \omega$, $a - \omega$ variabilis z, exprimentur sequenti modo:

$$\frac{\overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon}P^{\mathrm{I}} + \overset{r+1}{\alpha}\omega^{r}.\overset{r+1}{\varepsilon}P^{\mathrm{I}} + \overset{r+2}{\alpha}\omega^{r+1}.\overset{r+2}{\varepsilon}P^{\mathrm{I}} + \text{etc.}}{\overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon}Q^{\mathrm{I}} + \overset{r+1}{\alpha}\omega^{r}.\overset{r+1}{\varepsilon}Q^{\mathrm{I}} + \overset{r+2}{\alpha}\omega^{r+1}.\overset{r+2}{\varepsilon}Q^{\mathrm{I}} + \text{etc.}};$$

$$\frac{\pm\overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon}P^{\mathrm{I}} \mp \overset{r+1}{\alpha}\omega^{r}.\overset{r+1}{\varepsilon}Q^{\mathrm{I}} \pm \overset{r+2}{\alpha}\omega^{r+1}.\overset{r+2}{\varepsilon}P^{\mathrm{I}} \mp \text{etc.}}{\pm\overset{r}{\alpha}\omega^{r-1}.\overset{r}{\varepsilon}Q^{\mathrm{I}} \mp \overset{r+1}{\alpha}\omega^{r}.\overset{r+1}{\varepsilon}Q^{\mathrm{I}} \pm \overset{r+2}{\alpha}\omega^{r+1}.\overset{r+2}{\varepsilon}Q^{\mathrm{I}} \mp \text{etc.}}.$$

Hinc vero, denotante V valorem functionis $\frac{P}{Q}$ debitum valori $z = a$, sequitur ut in (n. 3.), pro certis coefficientibus $k, l, m, \ldots K, L, M$, etc. independentibus ab ω, et quapiam quantumcunque decrescente quantitate ω debere fieri

$$V > \frac{\overset{r}{\varepsilon}P^{\mathrm{I}}}{\overset{r}{\varepsilon}Q^{\mathrm{I}}} + k\omega + l\omega^2 + \text{etc.}; \text{ et } V < \frac{\overset{r}{\varepsilon}P^{\mathrm{I}}}{\overset{r}{\varepsilon}Q^{\mathrm{I}}} + K\omega + L\omega^2 + \text{etc.}:$$

igitur $V = \frac{\overset{r}{\varepsilon}P^{\mathrm{I}}}{\overset{r}{\varepsilon}Q^{\mathrm{I}}}$ (130. §.).

222. Corollarium 2.

Quamobrem, si petatur valor V, quem functio fracta $y=\frac{P}{Q}$ pro illo determinato valore a variabilis z debet habere, pro quo ea abit in $\frac{0}{0}$, sumantur quotientes $\frac{\delta P}{\delta Q}$, $\frac{\delta^2 P}{\delta^2 Q}$, $\frac{\delta^3 P}{\delta^3 Q}$, etc. posito $\delta z=1$, inchoandoque a primo fiat $z=a$: quoties enim aliquis horum quotorum pro $z=a$ in $\frac{0}{0}$ abiverit, nihil certi poterit inde colligi, sed quaeri debebit valor proxime sequentis quoti pro $z=a$: si autem ad quotum deveniatur, qui pro $z=a$ aut fiat aequalis nihilo, evanescente solo numeratore, vel obtineat valorem indefinitum, evanescente solo denominatore, aut valorem quempiam determinatum induat; debebit etiam data functio y pro $z=a$ casu primo aequari nihilo, casu autem secundo et tertio eundem indefinitum, vel determinatum valorem habere (220. 221. §.).

223. Corollarium 3.

Dum datur functio $y=PR$ variabilis z, quae pro certo valore $z=a$, evadente $P=0$ et $R=\frac{1}{0}=\infty$, abit in $0.\infty$, poni potest $Q=\frac{1}{R}$, hinc $y=\frac{P}{Q}$, quo casu, pro $z=a$, abibit y in $\frac{0}{0}$: invento itaque valore functionis $y=\frac{P}{Q}$ pro $z=a$ per (222. §.), habebitur valor functionis $y=PR$ debitus valori $z=a$.

224. Corollarium 4.

Et si aliqua functio $\frac{q}{p}$ variabilis z pro certo valore $z=a$, pro quo abeant q, p in $\frac{1}{0}=\infty$, abeat in $\frac{\infty}{\infty}$; poterit sumi $P=\frac{1}{p}$, et $Q=\frac{1}{q}$, hinc $\frac{q}{p}=\frac{Q}{P}$, quae functio pro $z=a$ in $\frac{0}{0}$ debebit abire: determinato igitur valore functionis $\frac{P}{Q}$ pro $z=a$ per (222. §.), obtinebitur eo ipso etiam valor functionis $\frac{q}{p}$ pro $z=a$.

225. Problema.

Fractionem rationalem genuinam $\frac{M}{N}$ (10. 11. §.) *datis factoribus simplicibus denominatoris* N, *qui omnes sint inaequales, resolvere in totidem fractiones partiales; quarum summa exaequet datam fractionem* $\frac{M}{N}$.

Solutio.

Pro quolibet factore simplici formae generalis $p-qx$ denominatoris N determinetur una fractio partialis $\frac{A}{p-qx}$, ita ut ejus numerator A eum valorem habeat, quem quotus $-\frac{qM}{\partial N}$ pro $x=\frac{p}{q}$, sumto exponente differentiali $\partial x=1$, obtineret.

Demonstratio.

Cum $p-qx$ sit unus factor functionis N, dabitur alia functio integra et rationalis X, pro qua fiat $N=X(p-qx)$: poterit igitur functio $\frac{M}{N}$ pro quapiam functione integra et rationali Z resolvi in duas fractiones, ita ut sit

$$\frac{M}{N}=\frac{A}{p-qx}+\frac{Z}{X}=\frac{AX+Z(p-qx)}{N}.$$

Adeoque habebimus $M=AX+Z(p-qx)$, et $Z=\frac{M-AX}{p-qx}$: $p-qx$ debet ergo esse unus factor simplex functionis $M-AX$ (169. §.), et ideo $M-AX=0$ pro $p-qx=0$, seu $A=\frac{M}{X}=\frac{M(p-qx)}{N}$ pro $x=\frac{p}{q}$.

Verum pro $x=\frac{p}{q}$ fit $M(p-qx)=0$, et $N=X(p-qx)=0$: numerator A aequatur igitur ei determinato valori, quem quotus $\frac{M(p-qx)}{N}$ pro valore $x=\frac{p}{q}$, pro quo is abit in $\frac{0}{0}$, obtinet; hinc debet esse (220. §.)

A=

$$A = \frac{\varepsilon . M(p-qx)}{\varepsilon N} = \frac{(p-qx)\varepsilon M}{\varepsilon N} - \frac{qM}{\varepsilon N} \text{ pro } x = \frac{p}{q},$$

seu $A = \frac{-qM}{\varepsilon N}$ pro $x = \frac{p}{q}$, ob $\frac{(p-qx)\varepsilon M}{\varepsilon N} = 0$ pro $x = \frac{p}{q}$.

226. Corollarium 1.

Si functio integra et rationalis $a+bx^n$ dicatur habere unum factorem simplicem $e-fx$, cum sit $\varepsilon(a+bx^n) = nbx^{n-1}$, adeoque

$$\frac{-fx^m}{\varepsilon(a+bx^n)} = \frac{-fx^m}{nbx^{n-1}} = \frac{-f^{n-m}}{nbe^{n-m-1}} \text{ pro } x = \frac{e}{f};$$

obtinebimus per (225. §.) sequentem expressionem pro una fractionum partialium, in quas functio fracta rationalis $\frac{x^m}{a+bx^n}$, datis omnibus factoribus simplicibus denominatoris, potest resolvi, nimirum pro fractione partiali oriunda ex factore simplici $e-fx$ denominatoris $a+bx^n$

$$\frac{-f^{n-m}}{nb(e-fx)e^{n-m-1}}.$$

227. Corollarium 2.

Si factores denominatoris N sint quidem omnes inaequales, sed vel omnes quadratici formae generalis $\alpha+\beta x+\gamma x^2$, vel aliqui quadratici et alii simplices; reperietur fractio partialis oriunda ex quovis factore quadratico, si hic resolvatur in duos factores simplices, fractionesque partiales his factoribus competentes per (225. §.) determinatae in unam summam addantur.

228. Corollarium 3.

Sic si $d^2-c^2x^2 = (d+cx)(d-cx)$ dicatur esse unus factor quadraticus functionis $a+bx^n$, functioque fracta rationalis $\frac{x^m}{a+bx^n}$ in fractiones partiales resolvi debeat; fractio partialis oriunda ex illo factore quadratico erit per (225. 227. §.),

$$\frac{\mp c^{n-m}}{nb(d+cx)d^{n-m-1}} - \frac{c^{n-m}}{nb(d-cx)d^{n-m-1}} =$$

$$= \frac{c^{n-m}}{nbd^{n-m-1}} \cdot \frac{\mp d - d \pm cx - cx}{d^2-c^2x^2};$$

consequenter, prout est $n-m$ par aut impar numerus, erit ea fractio

$$\frac{c^{n-m}}{nbd^{n-m-1}} \cdot \frac{-2}{d^2-c^2x^2}, \quad \text{vel} \quad \frac{c^{n-m+1}}{nbd^{n-m-1}} \cdot \frac{-2x}{d^2-c^2x^2}.$$

229. Theo-

229. Theorema.

Pro $x = \operatorname{Cos}\varphi \pm \operatorname{Sin}\varphi\sqrt{-1}$ *et quovis numero integro positivo* n *debet esse* $x^n = \operatorname{Cos} n\varphi \pm \operatorname{Sin} n\varphi\sqrt{-1}$.

Demonstratio.

Est enim $x^2 = \operatorname{Cos}\varphi^2 \pm 2\operatorname{Cos}\varphi \operatorname{Sin}\varphi\sqrt{-1} - \operatorname{Sin}\varphi^2$: igitur per (5. §. 1. Schol.) $x^2 = \operatorname{Cos} 2\varphi \pm \operatorname{Sin} 2\varphi\sqrt{-1}$. Et si pro quocunque numero integro r esset $x^r = \operatorname{Cos} r\varphi \pm \operatorname{Sin} r\varphi\sqrt{-1}$, deberet eo ipso esse etiam $x^{r+1} = \operatorname{Cos} r\varphi \operatorname{Cos}\varphi \pm \operatorname{Cos}\varphi \operatorname{Sin} r\varphi\sqrt{-1} \pm \operatorname{Sin}\varphi \operatorname{Cos} r\varphi\sqrt{-1} - \operatorname{Sin}\varphi \operatorname{Sin} r\varphi = \operatorname{Cos}(r+1)\varphi \pm \operatorname{Sin}(r+1)\varphi\sqrt{-1}$ per (5. §. 1. Schol.). Hinc itaque ob (31. §.) necessario sequitur, quod erat damonstrandum.

230. Corollarium 1.

Si $p(\operatorname{Cos}\varphi + \operatorname{Sin}\varphi\sqrt{-1}) - qx$ et $p(\operatorname{Cos}\varphi - \operatorname{Sin}\varphi\sqrt{-1}) - qx$ sint duo factores simplices imaginarii functionis rationalis $c + ex^n$; debebit haec functio bis aequari nihilo, pro $x = \frac{p}{q}(\operatorname{Cos}\varphi + \operatorname{Sin}\varphi\sqrt{-1})$ et $x = \frac{p}{q}(\operatorname{Cos}\varphi - \operatorname{Sin}\varphi\sqrt{-1})$: erit igitur per (229. §.)

$$c + \frac{ep^n}{q^n}(\operatorname{Cos} n\varphi + \operatorname{Sin} n\varphi\sqrt{-1}) = 0;$$

$$c + \frac{ep^n}{q^n}(\operatorname{Cos} n\varphi - \operatorname{Sin} n\varphi\sqrt{-1}) = 0.$$

Hinc $2c + \frac{2ep^n}{q^n}\operatorname{Cos} n\varphi = 0$; et $\frac{2ep^n}{q^n}\operatorname{Sin} n\varphi\sqrt{-1} = 0$;

seu $c + \frac{ep^n}{q^n}\operatorname{Cos} n\varphi = 0$; et $\operatorname{Sin} n\varphi = 0$.

231. Corollarium 2.

Quidquid sit functio rationalis et integra $a + bx^n$, necesse est, ut pro quolibet ejus factore quadratico $p^2 - 2pqx\operatorname{Cos}\varphi + q^2x^2$ ob (172. 230. §.) fiat

$$\operatorname{Cos} n\varphi = \frac{-aq^n}{bp^n}; \text{ et } \operatorname{Sin} n\varphi = 0.$$

232. Problema.

Datam functionem integram rationalem $a^n + x^n$ *resolvere in suos factores.*

Solutio.

Solutio.

1. Exponens n est numerus par vel impar. Pro numero pari $n=2m$ habebit functio a^n+x^n omnes numero $n=2m$ factores simplices imaginarios; adeoque, quia quivis bini factores simplices inter se multiplicati dant unum factorem quadraticum; habebit illa hoc casu factores quadraticos numero $m=\frac{1}{2}n$.

2. Si autem exponens $n=2m+1$ est impar numerus; habet functio a^n+x^n unum factorem simplicem realem, et $n-1$ factores simplices imaginarios, ex quibus numero $m=\frac{n-1}{2}$ factores quadratici possunt produci.

3. Quilibet factorum quadraticorum functionis a^n+x^n (1) (2) exprimi potest formula generali $p^2-2pqx\,\mathrm{Cos}\,\varphi+q^2x^2$ (172. §.); et pro quolibet debebit arcus φ ita determinari, ut sit per (230. §.) $a^n+\frac{p^n}{q^n}\mathrm{Cos}\,n\varphi=0$, et $\mathrm{Sin}\,n\varphi=0$.

4. Iam vero erit $\mathrm{Sin}\,n\varphi=0$, si, denotante k quemcunque terminum progressionis 0, 1, 2, 3, 4, etc., fiat $n\varphi=2k\pi$, vel $n\varphi=(2k+1)\pi$ per (5. §. 1. Schol.): praecedens formula (3) semper ergo dabit unum factorem quadraticum functionis a^n+x^n, si fiat

$$\varphi=\frac{2k\pi}{n},\ \text{et}\ a^n+\frac{p^n}{q^n}\mathrm{Cos}\,2k\pi=0;$$

$$\text{vel}\ \ \varphi=\frac{2(k+1)\pi}{n},\ \text{et}\ a^n+\frac{p^n}{q^n}\mathrm{Cos}\,2(k+1)\pi=0.$$

5. Prima aequatio in (4) nequit subsistere: cum enim sit $\mathrm{Cos}\,2k\pi=1$; esset $a^n+\frac{p^n}{q^n}=0$, hinc $\frac{p}{q}=\sqrt[n]{-a^n}$ valoris vel imaginarii, vel realis negativi, prout est n numerus par vel impar, cum tamen $\frac{p}{q}$ in (3) debeat habere valorem realem positivum.

6. Hinc, ob $\mathrm{Cos}(2k+1)\pi=-1$ (5. §. 1. Schol.), sequitur, formulam in (3) pro quolibet valore $k=0, k=1, k=2, k=3, k=4$, et sic porro, daturam unum factorem quadraticum functionis a^n+x^n, si fiat $\varphi=\frac{(2k+1)\pi}{n}$, et $a^n-\frac{p^n}{q^n}=0$ ob (4) (5), adeoque $p=a$ pro $q=1$:

229. Theorema.

Pro $x = \mathrm{Cos}\,\varphi \pm \mathrm{Sin}\,\varphi\sqrt{-1}$ *et quovis numero integro positivo* n *debet esse* $x^n = \mathrm{Cos}\,n\,\varphi \pm \mathrm{Sin}\,n\,\varphi\sqrt{-1}$.

Demonstratio.

Est enim $x^2 = \mathrm{Cos}\,\varphi^2 \pm 2\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi\sqrt{-1} - \mathrm{Sin}\,\varphi^2$: igitur per (5. §. 1. Schol.) $x^2 = \mathrm{Cos}\,2\varphi \pm \mathrm{Sin}\,2\varphi\sqrt{-1}$. Et si pro quocunque numero integro r esset $x^r = \mathrm{Cos}\,r\,\varphi \pm \mathrm{Sin}\,r\,\varphi\sqrt{-1}$, deberet eo ipso esse etiam $x^{r+1} = \mathrm{Cos}\,r\,\varphi\,\mathrm{Cos}\,\varphi \pm \mathrm{Cos}\,\varphi\,\mathrm{Sin}\,r\,\varphi\sqrt{-1} \pm \mathrm{Sin}\,\varphi\,\mathrm{Cos}\,r\,\varphi\sqrt{-1} - \mathrm{Sin}\,\varphi\,\mathrm{Sin}\,r\,\varphi = \mathrm{Cos}(r+1)\varphi \pm \mathrm{Sin}(r+1)\varphi\sqrt{-1}$ per (5. §. 1. Schol.). Hinc itaque ob (31. §.) necessario sequitur, quod erat damonstrandum.

230. Corollarium 1.

Si $p(\mathrm{Cos}\,\varphi + \mathrm{Sin}\,\varphi\sqrt{-1}) - qx$ et $p(\mathrm{Cos}\,\varphi - \mathrm{Sin}\,\varphi\sqrt{-1}) - qx$ sint duo factores simplices imaginarii functionis rationalis $c + ex^n$; debebit haec functio bis aequari nihilo, pro $x = \frac{p}{q}(\mathrm{Cos}\,\varphi + \mathrm{Sin}\,\varphi\sqrt{-1})$ et $x = \frac{p}{q}(\mathrm{Cos}\,\varphi - \mathrm{Sin}\,\varphi\sqrt{-1})$: erit igitur per (229. §.)

$$c + \frac{ep^n}{q^n}(\mathrm{Cos}\,n\,\varphi + \mathrm{Sin}\,n\,\varphi\sqrt{-1}) = 0;$$

$$c + \frac{ep^n}{q^n}(\mathrm{Cos}\,n\,\varphi - \mathrm{Sin}\,n\,\varphi\sqrt{-1}) = 0.$$

Hinc $2c + \frac{2ep^n}{q^n}\mathrm{Cos}\,n\,\varphi = 0$; et $\frac{2ep^n}{q^n}\mathrm{Sin}\,n\,\varphi\sqrt{-1} = 0$;

seu $c + \frac{ep^n}{q^n}\mathrm{Cos}\,n\,\varphi = 0$; et $\mathrm{Sin}\,n\,\varphi = 0$.

231. Corollarium 2.

Quidquid sit functio rationalis et integra $a + bx^n$, necesse est, ut pro quolibet ejus factore quadratico $p^2 - 2pqx\,\mathrm{Cos}\,\varphi + q^2x^2$ ob (172. 230. §.) fiat

$$\mathrm{Cos}\,n\,\varphi = \frac{-aq^n}{bp^n}; \text{ et } \mathrm{Sin}\,n\,\varphi = 0.$$

232. Problema.

Datam functionem integram rationalem $a^n + x^n$ *resolvere in suos factores.*

Solutio.

Solutio.

1. Exponens n est numerus par vel impar. Pro numero pari $n = 2m$ habebit functio $a^n + x^n$ omnes numero $n = 2m$ factores simplices imaginarios; adeoque, quia quivis bini factores simplices inter se multiplicati dant unum factorem quadraticum; habebit illa hoc casu factores quadraticos numero $m = \frac{1}{2} n$.

2. Si autem exponens $n = 2m + 1$ est impar numerus; habet functio $a^n + x^n$ unum factorem simplicem realem, et $n - 1$ factores simplices imaginarios, ex quibus numero $m = \frac{n-1}{2}$ factores quadratici possunt produci.

3. Quilibet factorum quadraticorum functionis $a^n + x^n$ (1) (2) exprimi potest formula generali $p^2 - 2pqx \operatorname{Cos}\varphi + q^2 x^2$ (172. §.); et pro quolibet debebit arcus φ ita determinari, ut sit per (230. §.) $a^n + \frac{p^n}{q^n} \operatorname{Cos} n\varphi = 0$, et $\operatorname{Sin} n\varphi = 0$.

4. Iam vero erit $\operatorname{Sin} n\varphi = 0$, si, denotante k quemcunque terminum progressionis 0, 1, 2, 3, 4, etc., fiat $n\varphi = 2k\pi$, vel $n\varphi = (2k+1)\pi$ per (5. §. 1. Schol.): praecedens formula (3) semper ergo dabit unum factorem quadraticum functionis $a^n + x^n$, si fiat

$$\varphi = \frac{2k\pi}{n}, \text{ et } a^n + \frac{p^n}{q^n} \operatorname{Cos} 2k\pi = 0;$$

$$\text{vel } \varphi = \frac{2(k+1)\pi}{n}, \text{ et } a^n + \frac{p^n}{q^n} \operatorname{Cos} 2(k+1)\pi = 0.$$

5. Prima aequatio in (4) nequit subsistere: cum enim sit $\operatorname{Cos} 2k\pi = 1$; esset $a^n + \frac{p^n}{q^n} = 0$, hinc $\frac{p}{q} = \sqrt[n]{-a^n}$ valoris vel imaginarii, vel realis negativi, prout est n numerus par vel impar, cum tamen $\frac{p}{q}$ in (3) debeat habere valorem realem positivum.

6. Hinc, ob $\operatorname{Cos}(2k+1)\pi = -1$ (5. §. 1. Schol.), sequitur, formulam in (3) pro quolibet valore $k = 0, k = 1, k = 2, k = 3, k = 4$, et sic porro, daturam unum factorem quadraticum functionis $a^n + x^n$, si fiat $\varphi = \frac{(2k+1)\pi}{n}$, et $a^n - \frac{p^n}{q^n} = 0$ ob (4) (5), adeoque $p = a$ pro $q = 1$:

 his

his ergo valoribus in (3) substitutis, obtinebitur sequens generalis expressio omnium factorum quadraticorum functionis $a^n + x^n$.

$$a^2 - 2ax \operatorname{Cos}\frac{(2k+1)\pi}{n} + x^2.$$

7. Quamobrem, si in (6) loco k successive substituantur numeri 0, 1, 2, 3, 4, 5, et ita porro quamdiu est $2k+1 < n$, obtinebuntur omnes factores quadratici functionis $a^n + x^n$, ad quos praeterea eo casu, quo n fuerit numerus impar, factor simplex $a+x$ debebit accedere (1) (2).

Exempla.

$$a^3+x^3 = (a^2 - 2ax\operatorname{Cos}\frac{\pi}{3} + x^2)(a+x).$$

$$a^4+x^4 = (a^2 - 2ax\operatorname{Cos}\frac{\pi}{4} + x^2)(a^2 - 2ax\operatorname{Cos}\frac{3\pi}{4} + x^2).$$

$$a^5+x^5 = (a^2 - 2ax\operatorname{Cos}\frac{\pi}{5} + x^2)(a^2 - 2ax\operatorname{Cos}\frac{3\pi}{5} + x^2)(a+x).$$

$$a^6+x^6 = (a^2 - 2ax\operatorname{Cos}\frac{\pi}{6} + x^2)(a^2 - 2ax\operatorname{Cos}\frac{3\pi}{6} + x^2)(a^2 - 2ax\operatorname{Cos}\frac{5\pi}{6} + x^2).$$

Scholion.

Sed, quaeret tyro, cur illi duntaxat numeri impares loco $2k+1$ in (7), qui numero n minores sunt, substitui praecipiuntur? dabit utique formula generalis in (6) etiam pro quovis alio majori numero impari $2k+1$ unum factorem quadraticum functionis $a^n + x^n$, ob (4). Verum id quoque certum est, factores quadraticos, qui pro numeris imparibus $2k+1 > n$ ex (6) possunt elici, ab iis, qui pro numeris $2k+1 < n$ indidem eliciuntur, haud esse distinctos. Sit enim generatim $2k+1 = 2rn \pm \alpha$ numerus major numero n, ita ut r quemlibet terminum seriei 1, 2, 3, 4, etc. possit denotare, α vero denotet numerum imparem minorem numero n: dico ex (6) pro $2k+1 = 2rn \pm \alpha$ eundem sequi factorem, qui pro $2k+1 = \alpha$ obtineretur. Est enim $\operatorname{Cos}\frac{(2rn \pm \alpha)\pi}{n} = \operatorname{Cos}(2r\pi \pm \frac{\alpha\pi}{n}) = \operatorname{Cos} 2r\pi$ $\operatorname{Cos}\frac{\alpha\pi}{n} \mp \operatorname{Sin} 2r\pi \operatorname{Sin}\frac{\alpha\pi}{n} = \operatorname{Cos}\frac{\alpha\pi}{n}$ (5. §. 1. Schol.)

233. Problema.

Functionem integram rationalem $a^n - x^n$ resolvere in suos factores.

Solutio.

Solutio.

1. Pro exponente pari n habebit functio $a^n - x^n$ duos factores simplices reales, et $n=2$ factores simplices imaginarios, et ex his omnibus nascentur numero $\frac{n-2}{2}$ factores quadratici.

2. Et pro numero impari n habebit functio $a^n - x^n$ unum factorem simplicem realem, cum $n-1$ factoribus simplicibus imaginariis, qui simul dabunt numero $\frac{n-1}{2}$ factores quadraticos.

3. Quilibet factor quadraticus functionis $a^n - x^n$ (1) (2) obtinebitur ex $p^2 - 2pqx\,\mathrm{Cos}\,\varphi + q^2x^2$ (172. §.), si pro quocunque termino progressionis arithmeticae 1, 2, 3, 4, 5, 6, etc. loco k assumto fiat per (230. §.) et (5. §. 1. Schol.).

$$\varphi = \frac{2k\pi}{n},\ a^n - \frac{p^n}{q^n} = 0;\ \text{et Sin}\,2k\pi = 0;$$

$$\text{vel}\ \varphi = \frac{(2k+1)\pi}{n};\ a^n + \frac{p^n}{q^n} = 0;\ \text{et Sin}(2k+1)\pi = 0.$$

4. Cum vero secunda conditio in (3) absurda sit, ut in (232. §. n.5.); certum est, formulam generalem factoris quadratici in (3) daturam quemvis factorem quadraticum functionis $a^n - x^n$, si in illa ponatur $\varphi = \frac{2k\pi}{n}$, et $a^n - \frac{p^n}{q^n} = 0$, hinc $a = p$ pro $q = 1$, denotante k quemcunque terminum seriei arithmeticae 1, 2, 3, 4, etc.: his igitur valoribus in (3) substitutis obtinebimus sequentem expressionem generalem omnium factorum quadraticorum functionis $a^n - x^n$.

$$a^2 - 2ax\,\mathrm{Cos}\frac{2k\pi}{n} + x^2.$$

5. Quare, si in hac formula (4) successive ponas $k=1$, $k=2$, $k=3$, $k=4$, et sic porro quamdiu est $2k<n$; derivabis inde omnes factores quadraticos functionis $a^n - x^n$, ad quos praeterea accedet factor simplex $a-x$, vel quadraticus $a^2 - x^2 = (a-x)(a+x)$, prout fuerit n numerus impar, vel par (2) (1).

Exempla.

$$a^3 - x^3 = (a^2 - 2ax\,\mathrm{Cos}\frac{2\pi}{3} + x^2)(a-x).$$

$$a^4 - x^4 = (a^2 - 2ax\,\mathrm{Cos}\frac{2\pi}{4} + x^2)(a^2 - x^2) = (a^2 + x^2)(a^2 - x^2).$$

$$a^5-x^5=(a^2-2ax\,\mathrm{Cos}\frac{2\pi}{5}+x^2)(a^2-2ax\,\mathrm{Cos}\frac{4\pi}{5}+x^2)(a-x).$$

$$a^6-x^6=(a^2-2ax\,\mathrm{Cos}\frac{2\pi}{6}+x^2)(a^2-2ax\,\mathrm{Cos}\frac{4\pi}{6}+x^2)(a^2-x^2).$$

Scholion.

Ratio autem, cur loco $2k$ illi tantum numeri pares, qui numero n minores sunt, in (4) substitui praecipiuntur, est eadem, quam jam pro casu simili in (232. §. Schol.) adduximus. Omnis enim numerus par major numero n per $2rn\pm 2\alpha$ potest designari, ita ut, denotante 2α quemcunque numerum parem minorem numero n, coefficiens r quemlibet terminum seriei 1, 2, 3, 4, etc. possit indicare: facile vero ostenditur, formulam in (4) eosdem pro $2k=2rn\pm 2\alpha$, et $2k=2\alpha$ factores dare. Nam est $\mathrm{Cos}\frac{(2rn\pm 2\alpha)\pi}{n}=\mathrm{Cos}(2r\pi\pm\frac{2\alpha\pi}{n})=\mathrm{Cos}\,2r\pi\,\mathrm{Cos}\frac{2\alpha\pi}{n}$ $\mp\mathrm{Sin}\,2r\pi\,\mathrm{Sin}\frac{2\alpha\pi}{n}=\mathrm{Cos}\frac{2\alpha\pi}{n}$ (5. §. 1. Schol.)

234. Problema.

Unus factor quadraticus functionis integrae et rationalis $a+bx^n$ *est* $p^2-2pqx\,\mathrm{Cos}\varphi+q^2x^2$; *determinare fractionem partialem ex ipso oriundam, data functione fracta rationali* $\frac{x^m}{a+bx^n}$ *in fractiones partiales resolvenda.*

Solutio.

1. Factores simplices dati factoris quadratici sunt $qx-p(\mathrm{Cos}\varphi+\mathrm{Sin}\varphi\sqrt{-1})$, $qx-p(\mathrm{Cos}\varphi-\mathrm{Sin}\varphi\sqrt{-1})$ per (172. §.): quodsi ergo numeratores fractionum partialium his factoribus simplicibus debitarum vocentur A, B; erit per (227. §.) fractio partialis oriunda ex dato factore quadratico

$$\frac{A}{qx-p(\mathrm{Cos}\varphi+\mathrm{Sin}\varphi\sqrt{-1})}+\frac{B}{qx-p(\mathrm{Cos}\varphi-\mathrm{Sin}\varphi\sqrt{-1})}=$$
$$=\frac{q(A+B)x+(A-B)p\,\mathrm{Sin}\varphi\sqrt{-1}-(A+B)p\,\mathrm{Cos}\varphi}{p^2-2pqx\,\mathrm{Cos}\varphi+q^2x^2}.$$

2. Iam sit $M=x^m$, $N=a+bx^n$, hinc $\varepsilon N=nbx^{n-1}\varepsilon x$, et per (229. §.)

$$x^m=\frac{p^m}{q^m}(\mathrm{Cos}\,m\varphi\pm\mathrm{Sin}\,m\varphi\sqrt{-1});$$

$$x^{n-1}=\frac{p^{n-1}}{q^{n-1}}(\mathrm{Cos}(n-1)\varphi\pm\mathrm{Sin}(n-1)\varphi\sqrt{-1});$$

pro $x=\frac{p}{q}(\mathrm{Cos}\varphi\pm\mathrm{Sin}\varphi\sqrt{-1})$;

Per

Per haec determinantur valores numeratorum A, B fractionum partialium in (1) secundum (225. §.), nimirum:

$$A = \frac{q^{n-m}}{n b p^{n-m-1}} \cdot \frac{\operatorname{Cos} m\varphi + \operatorname{Sin} m\varphi \sqrt{-1}}{\operatorname{Cos}(n-1)\varphi + \operatorname{Sin}(n-1)\varphi \sqrt{-1}};$$

$$B = \frac{q^{n-m}}{n b p^{n-m-1}} \cdot \frac{\operatorname{Cos} m\varphi - \operatorname{Sin} m\varphi \sqrt{-1}}{\operatorname{Cos}(n-1)\varphi - \operatorname{Sin}(n-1)\varphi \sqrt{-1}}.$$

Est autem per (5. §. 1. Schol. et 231. §.).

$$\operatorname{Cos}(n-1)\varphi = \operatorname{Cos} n\varphi \operatorname{Cos}\varphi + \operatorname{Sin} n\varphi \operatorname{Sin}\varphi = -\frac{a q^n}{b p^n} \operatorname{Cos}\varphi;$$

$$\operatorname{Sin}(n-1)\varphi = \operatorname{Sin} n\varphi \operatorname{Cos}\varphi - \operatorname{Cos} n\varphi \operatorname{Sin}\varphi = \frac{a q^n}{b p^n} \operatorname{Sin}\varphi:$$

adeoque est etiam

$$A = \frac{p^{m+1}}{n a q^m} \cdot \frac{\operatorname{Cos} m\varphi + \operatorname{Sin} m\varphi \sqrt{-1}}{\operatorname{Sin}\varphi\sqrt{-1} - \operatorname{Cos}\varphi};$$

$$B = \frac{p^{m+1}}{n a q^m} \cdot \frac{\operatorname{Sin} m\varphi \sqrt{-1} - \operatorname{Cos} m\varphi}{\operatorname{Sin}\varphi\sqrt{-1} + \operatorname{Cos}\varphi}.$$

Quare addendo, et subtrahendo erit

ob $(\operatorname{Sin}\varphi\sqrt{-1} - \operatorname{Cos}\varphi)(\operatorname{Sin}\varphi\sqrt{-1} + \operatorname{Cos}\varphi) = -(\operatorname{Sin}\varphi^2 + \operatorname{Cos}\varphi^2) = -1.$

$$A + B = -\frac{2 p^{m+1}}{n a q^m} \operatorname{Cos}(m+1)\varphi;$$

$$A - B = -\frac{2 p^{m+1}}{n a q^m} \operatorname{Sin}(m+1)\varphi\sqrt{-1}.$$

Et pro his valoribus in (1) substitutis obtinebitur sequens simplicissima expressio quaesitae fractionis partialis.

$$\frac{2 p^{m+1}}{n a q^m} \cdot \frac{p \operatorname{Cos} m\varphi - q x \operatorname{Cos}(m+1)\varphi}{p^2 - 2 p q x \operatorname{Cos}\varphi + q^2 x^2}.$$

Scholion.

Notatu digna est methodus quasvis functiones fractas rationales, datis factoribus simplicibus et quadraticis denominatorum, resolvendi in fractiones partiales ope coefficientium indeterminatorum.

1. Si proponatur fractio $\frac{M}{N}$ resolvenda in fractiones partiales, datis factoribus simplicibus formae $p - qx$, et quadraticis formae $\alpha + \beta x + \gamma x^2$ denominatoris N; sumatur pro quolibet factore simplici una fractio partialis $\frac{A}{p - qx}$, et pro quolibet factore quadratico accipiatur una fractio formae $\frac{A + Bx}{\alpha + \beta x + \gamma x^2}$, literis A, B indeterminatas interea quantitates denotantibus.

 Tum

Tum reducantur omnes hae fractiones partiales ad communem denominatorem aequalem producto N ex illarum denominatoribus; erit summa omnium numeratorum divisa per N aequalis datae functioni $\frac{M}{N}$. Quodsi igitur summa omnium numeratorum aequetur dato numeratori M; obtinebitur aequatio, ex qua valores pro A, B, etc. secundum principia primi capitis poterunt determinari.

2. Eadem prorsus methodo resolvetur quaelibet fractio $\frac{M}{X^n}$ in fractiones partiales, dum denominator fractionis resolvendae aequatur alicui potentiae *ntae* aut factoris simplicis $X = p - qx$, aut quadratici $X = \alpha + \beta x + \gamma x^2$: nimirum, sumtis interea quantitatibus indeterminatis A, B, C, D, E, F, G, etc., ponatur

$$\frac{M}{X^n} = \frac{A}{X^n} + \frac{B}{X^{n-1}} + \frac{C}{X^{n-2}} + \text{---} + \frac{K}{X^2} + \frac{L}{X},$$

$$\text{vel } \frac{M}{X^n} = \frac{A+Bx}{X^n} + \frac{C+Dx}{X^{n-1}} + \text{---} + \frac{U+Vx}{X},$$

prout X exhibet factorem simplicem, vel quadraticum, tum quaerantur valores pro A, B, C, etc., ut in (1).

3. Si denique denominator N fractionis resolvendae $\frac{M}{N}$ inaequales aliquos, et alios aequales factores simplices vel quadraticos, aut illos et hos habeat; habebitur casus ex praecedentibus casibus (1) (2) compositus, ad quem idcirco eadem methodus poterit adplicari.

4. Licebit demum hanc methodum ad quosvis factores polynomios extendere: si nimirum denominator N fractionis resolvendae $\frac{M}{N}$ habeat factores polynomios formae generalis $a + bx + cx^2 + \text{---} + mx^r$; semper poterit determinari una fractio partialis debita tali factori polynomio, modo ejus numeratori tribuatur forma $\alpha + \beta x + \gamma x^2 + \text{---} \mu x^{r-1}$ functionis ordinis uno gradu inferioris.

ELEMEN-

ELEMENTA

CALCULI INTEGRALIS.

CAPUT IV.

DE PRAECEPTIS FUNDAMENTALIBUS CALCULI INTEGRALIS, INTEGRATIONEQUE FUNCTIONUM RATIONALIUM.

235. Definitio.

Integrale dati exponentis differentialis eſt functio, cujus ratio differentialis habet exponentem dato aequalem; unde, quid ſit datum exponentem differentialem *integrare*, per ſe intelligitur: praecepta in hunc finem docet *Calculus integralis.* Signum autem integrationis erit $\int$, praefigendum exponenti differentiali, cujus integrale fuerit deſideratum: ſic $\int X\partial x$ indicabit integrale exponentis differentialis $X\partial x$.

236. Corollarium 1.

Integrale dati exponentis differentialis $\partial y = X\partial x$ poteſt eſſe functio $y = Z + C$ compoſita ex aliqua functione Z variabilis x, et quantitate conſtante C, quarum poſterior nullo pacto immediate ex ipſo exponente differentiali poteſt colligi (91. §.).

237. Corollarium 2.

Quamobrem praecepta calculi integralis eo tendunt, ut per ea ex datis exponentibus differentialibus ∂y partes variabiles illorum integralium y utpote cum iis certa lege connexae (83. §.) poſſint derivari: plerumque tamen, inventa parte variabili, puta Z, alicujus integralis $\int\partial y = y$, ſcribemus $y = Z + C$, litera C quantitatem conſtantem deſignaturi, quae fors addi debebit parti variabili Z, ut perfectum completumque integrale habeatur.

238. Corollarium 3.

Facile vero, inventa parte variabili Z, pars conſtans C certi integralis $y = Z + C$ in actuali calculi differentio-integralis adplicatione determinatur: plerumque enim ex natura quanti, pro quo expreſſio analytica $y = Z + C$

quaeritur, elucet, quemnam determinatum valorem V functio y pro certo valore v variabilis x debet induere, quo fit, ut, si pro $x = v$ pars variabilis Z valorem determinatum U obtineat, fit $U + C = V$, hinc quantitas constans $C = V - U$, et integrale ipsum $y = Z + V - U$.

239. Corollarium 4.

Methodus universalis explorandi, utrum pars variabilis Z alicujus integralis $\int X \varepsilon x = Z + C$ rite sit determinata, consistit in differentiatione inventi integralis $Z + C$: si enim exponens differentialis $\varepsilon(Z+C) = \varepsilon Z$ aequetur dato $X \varepsilon x$; certi erimus, integrale $\int X \varepsilon x = Z + C$ esse legitimum, ita ut nullum aliud exponenti differentiali $X \varepsilon x$ possit respondere, cujus pars variabilis, non tantum forma, sed etiam magnitudine a parte variabili Z differat. (235. 96. §.).

240. Corollarium 5.

Hinc et ex (115. - - - 123. §.) colligemus sequentia integralia, quae instar totidem formularum fundamentalium calculi integralis possunt spectari.

1. $\int \varepsilon\varphi \,\mathrm{Cos}\,\varphi = \mathrm{Sin}\,\varphi + C.$
2. $\int -\varepsilon\varphi \,\mathrm{Sin}\,\varphi = \mathrm{Cosin}\,\varphi + C.$
3. $\int \varepsilon\varphi \,\mathrm{Sin}\,\varphi = \mathrm{Sin\,v}\,\varphi + C.$
4. $\int -\varepsilon\varphi \,\mathrm{Cos}\,\varphi = \mathrm{Cos\,v}\,\varphi + C.$
5. $\int \varepsilon\varphi \,\mathrm{Sec}\,\varphi^2 = \mathrm{Tang}\,\varphi + C.$
6. $\int -\varepsilon\varphi \,\mathrm{Cosec}\,\varphi^2 = \mathrm{Cot}\,\varphi + C.$
7. $\int \varepsilon\varphi \,\mathrm{Sec}\,\varphi \,\mathrm{Tang}\,\varphi = \mathrm{Sec}\,\varphi + C.$
8. $\int -\varepsilon\varphi \,\mathrm{Cosec}\,\varphi \,\mathrm{Cot}\,\varphi = \mathrm{Cosec}\,\varphi + C.$
9. $\int \frac{\varepsilon z}{\sqrt{(1 - z^2)}} = \mathrm{Arc\,Sin}\,z + C.$
10. $\int \frac{-\varepsilon z}{\sqrt{(1 - z^2)}} = \mathrm{Arc\,Cos}\,z + C.$
11. $\int \frac{\varepsilon z}{\sqrt{(2z - z^2)}} = \mathrm{Arc\,Sin\,v}\,z + C.$
12. $\int \frac{-\varepsilon z}{\sqrt{(2z - z^2)}} = \mathrm{Arc\,Cos\,v}\,z + C.$
13. $\int \frac{\varepsilon z}{1 + z^2} = \mathrm{Arc\,Tang}\,z + C.$
14. $\int \frac{-\varepsilon z}{1 + z^2} = \mathrm{Arc\,Cot}\,z + C.$
15. $\int \frac{\varepsilon z}{z\sqrt{(z^2 - 1)}} = \mathrm{Arc\,Sec}\,z + C.$
16. $\int \frac{-\varepsilon z}{z\sqrt{(z^2 - 1)}} = \mathrm{Arc\,Cosec}\,z + C.$

241. Corollarium 6.

Sic quoque ex (239. 90. §.) sequitur, integrale cujusvis exponentis differentialis $A \varepsilon Z$, determinati per productum ex exponente differentiali εZ certae functionis Z in quantitatem constantem A, aequari producto ex integrali hujus exponentis in eandem quantitatem constantem, seu esse $\int A \varepsilon Z = A \int \varepsilon Z$.

242. Co-

242. Corollarium 7.

Ob (239. 92. §.) autem erit integrale exponentis differentialis $\varepsilon y = \varepsilon P + \varepsilon Q + \varepsilon R + - - - + \varepsilon Z$ compositi ex pluribus exponentibus differentialibus aequale summae integralium singulorum exponentium illum componentium, nimirum $y = \int(\varepsilon P + \varepsilon Q + \varepsilon R + - - - + \varepsilon Z) = \int \varepsilon P + \int \varepsilon Q + \int \varepsilon R + - - - + \int \varepsilon Z$.

243. Corollarium 8.

Si in aliquod integrale $\int X \varepsilon x$ quantitas quaepiam constans, per additionem aut subtractionem, ingressa fuerit, poterit ea negligi, atque ad constantem indeterminatam C referri (237. §.); consequenter in specie loco $\int X \varepsilon x = U + \log \frac{m}{n} Z + C$, cum sit $l \frac{m}{n} Z = lZ + l \frac{m}{n}$, poterit sumi $\int X \varepsilon x = U + lZ + C$, quamdiu partis duntaxat variabilis ratio habetur.

244. Corollarium 9.

Dum integralia per arcus circuli determinantur, loco arcuum negativorum substitui possunt arcus positivi, modo illorum sinus, cosinus, sinus versi, cosinus versi, etc. cum signis contrariis sumantur, vel in cosinus, sinus, cosinus versos, sinus versos, etc. convertantur (239. 240. §.).

245. Problema.

Integrare exponentem differentialem $\varepsilon y = \frac{\varepsilon z}{z}$ fractum, cujus numerator aequatur exponenti differentiali denominatoris z, quidquid sit iste, aliqua variabilis absoluta, aut quaecunque functio certae variabilis absolutae (4. §.).

Solutio.

Sumatur pro integrali logarithmus naturalis denominatoris, seu fiat $y = \int \frac{\varepsilon z}{z} = lz + C$ (239. 110. §.).

246. Corollarium.

Si $y = \frac{AZ^n}{n} + C$ dicatur esse integrale alicujus exponentis differentialis εy, quod pro $n = 0$ abit in $\frac{AZ^0}{0} + C$; erit $A \log Z + C$ genuinus valor ejus integralis debitus valori $n = 0$: cum enim debeat esse $\varepsilon y = \varepsilon \left(\frac{AZ^n}{n} + C\right) = AZ^{n-1} \varepsilon Z$ (239. §.); debet utique datum integrale pro $n = 0$ poni $= \int AZ^{0-1} \varepsilon Z = \int \frac{A \varepsilon Z}{Z} = AlZ$ (245. 241. §.).

 247. Pro-

247. Problema.

Integrare exponentes differentiales formula generali $AZ^m \partial Z$ comprehensos, quidquid sit z, variabilis absoluta, aut aliqua functio certae variabilis absolutae.

Solutio.

Potentia variabilis z uno gradu altior divisa per suum exponentem $m+1$ ductaque in datum factorem constantem A dabit integrale quaesitum, seu erit (239. 104. §.)

$$\int A z^m \partial z = \frac{A z^{m+1}}{m+1} + C.$$

248. Corollarium 1.

Per (247. §.) integrabitur quivis exponens differentialis formae $X \partial x (a+Y)^p$, si functio $X \partial x$ extra vinculum aequetur exponenti differentiali functionis $a+Y$ sub vinculo: posito enim $z = a+Y$, erit $\partial z = \partial(a+Y) = X \partial x$; hinc $\int X \partial x (a+Y)^p = \int \partial z z^p = \frac{z^{p+1}}{p+1} = \frac{(a+Y)^{p+1}}{p+1} + C.$

249. Corollarium 2.

Si detur exponens differentialis formae $\partial y = A z^a \partial z + B z^b \partial z + C z^c \partial z +$ etc. $= (A z^a + B z^b + C z^c +$ etc.$) \partial z$; erit illius integrale, nulla quantitatis constantis ratione habita, $y = \frac{A z^{a+1}}{a+1} + \frac{B z^{b+1}}{b+1} + \frac{C z^{c+1}}{c+1} +$ etc. (247. 242. §.).

250. Corollarium 3.

Hinc elucet, quemlibet exponentem differentialem $\partial y = P^p Q^q R^r - - - X^u \partial z$ esse perfecte integrabilem, si P, Q, R, - - - X quascunque functiones variabilis z formae generalis $K z^k + L z^l + M z^m +$ etc., et p, q, r, - - - u numeros integros positivos denotent: semper enim poterit exponens ∂y explicari per seriem formae $A z^a + B z^b + C z^c +$ etc. ductam in ∂z constantemque terminis numero finitis, utprimum numerus terminorum cujuslibet functionis P, Q, R, - - - X seorsim finitus est.

251. Corollarium 4.

Quamobrem nulla poterit assignari functio integra et rationalis $X \partial z$ variabilis z, cujus integrale perfectum per principia praecedentia determinari nequeat.

252. Problema.

Integrare exponentes differentiales fractos, formulis $\frac{A\,\varepsilon x}{p+qx}$, $\frac{A\,\varepsilon x}{g\pm ex^2}$ *comprehensos.*

Solutio.

Pro $z=p+qx$ fiet $\frac{A\,\varepsilon x}{p+qx}=\frac{A}{q}\cdot\frac{\varepsilon z}{z}$, unde per (241. 245. §.) obtinetur sequens integrale (I). Porro est $\frac{A\,\varepsilon x}{g-ex^2}=\frac{A\,\varepsilon x}{2g+2x\sqrt{ge}}+\frac{A\,\varepsilon x}{2g-2x\sqrt{ge}}$, unde, integrando ope inventae formulae (I), per (242. §.) invenietur sequens integrale (II). Denique pro $z=\frac{x\sqrt{e}}{\sqrt{g}}$ fiet $\frac{A\,\varepsilon x}{g+ex^2}=\frac{A}{\sqrt{ge}}\cdot\frac{\varepsilon z}{1+z^2}$; et hinc per (241. 240. §.) reperitur sequens integrale (III).

$$\text{I.}\ \int\frac{A\,\varepsilon x}{p+qx}=\frac{A}{q}\,l(p+qx)+C.$$

$$\text{II.}\ \int\frac{A\,\varepsilon x}{g-ex^2}=\frac{A}{2\sqrt{ge}}\,l\left(\frac{\sqrt{g}+x\sqrt{e}}{\sqrt{g}-x\sqrt{e}}\right)+C.$$

$$\text{III.}\ \int\frac{A\,\varepsilon x}{g+ex^2}=\frac{A}{\sqrt{ge}}\ \text{Arc Tang}\ \frac{x\sqrt{e}}{\sqrt{g}}+C.$$

253. Problema.

Integrare exponentes differentiales formula generali $\frac{\varepsilon x}{a+bx\pm cx^2}$ *comprehensos.*

Solutio.

Pro $z=x\pm\frac{b}{2c}$, hinc $\varepsilon z=\varepsilon x$ transformabitur datus exponens in $\frac{\varepsilon x}{a+bx\pm cx^2}=\frac{\varepsilon z}{\frac{\pm 4ac-b^2}{\pm 4c}\pm cz^2}$: igitur integrando per (252. §.) obtinebimus sequentia integralia.

$$\int\frac{\varepsilon x}{a+bx+cx^2}=\frac{2}{\sqrt{(4ac-b^2)}}\,\text{Arc Tang}\,\frac{2cx+b}{\sqrt{(4ac-b^2)}}+C.$$

$$\int\frac{\varepsilon x}{a+bx-cx^2}=\frac{1}{\sqrt{(4ac+b^2)}}\,l\left(\frac{\sqrt{(4ac+b^2)}+2cx-b}{\sqrt{(4ac+b^2)}-2cx+b}\right)+C.$$

In adplicatione vero harum formularum ad casus speciales quantitates imaginarias, quas prima formula, existente $b^2 > 4ac$, dederit, vitabimus, si in dato exponente differentiali signa mutemus in contraria, cum dein ope secundae formulae integremus. Sic ope primae formulae reperietur $\int\frac{\partial x}{1-3x+x^2} = \frac{2}{\sqrt{-5}}\,\mathrm{Arc\,Tang}\,\frac{2x-3}{\sqrt{-5}} + C$: secunda autem formula dabit $\int\frac{-\partial x}{-1+3x-x^2} = C - \frac{1}{\sqrt{5}}\,l\left(\frac{\sqrt{5}+2x-3}{\sqrt{5}-2x+3}\right)$.

254. Corollarium 1.

Cum sit $\partial(a+bx\pm cx^2) = b\partial x \pm 2cx\partial x$; erit $\frac{x\partial x}{a+bx\pm cx^2} = \pm\frac{1}{2c}\cdot\frac{\partial(a+bx\pm cx^2)}{a+bx\pm cx^2} \mp \frac{b}{2c}\cdot\frac{\partial x}{a+bx\pm cx^2}$: integrando itaque per (242. 241. 245. 253. §.) inveniemus sequentes formulas.

$$\int\frac{x\partial x}{a+bx+cx^2} = C + \frac{1}{2c}\,l(a+bx+cx^2) - \frac{b}{c\sqrt{(4ac-b^2)}}\,\mathrm{Arc Tang}\frac{2cx+b}{\sqrt{(4ac-b^2)}}.$$

$$\int\frac{x\partial x}{a+bx-cx^2} = C - \frac{1}{2c}\,l(a+bx-cx^2) + \frac{b}{2c\sqrt{(4ac+b^2)}}\,l\left(\frac{\sqrt{(4ac+b^2)}+2cx-b}{\sqrt{(4ac+b^2)}-2cx+b}\right).$$

255. Corollarium 2.

Porro est $\frac{(A+Bx)\partial x}{a+bx\pm cx^2} = \frac{A\partial x}{a+bx\pm cx^2} + \frac{Bx\partial x}{a+bx\pm cx^2}$: per (242. 241. 253. 254. §.) obtinebimus ergo sequentia integralia.

$$\int\frac{(A+Bx)\partial x}{a+bx+cx^2} = C + \frac{B}{2c}\,l(a+bx+cx^2) + \frac{2Ac-Bb}{c\sqrt{(4ac-b^2)}}\,\mathrm{Arc Tang}\,\frac{2cx+b}{\sqrt{(4ac-b^2)}}.$$

$$\int\frac{(A+Bx)\partial x}{a+bx-cx^2} = C - \frac{B}{2c}\,l(a+bx-cx^2) + \frac{2Ac+Bb}{2c\sqrt{(4ac+b^2)}}\,l\left(\frac{\sqrt{(4ac+b^2)}+2cx-b}{\sqrt{(4ac+b^2)}-2cx+b}\right).$$

256. Corollarium 3.

Formula $\frac{A\partial x}{x(a+bx\pm cx^2)} = \frac{A}{a}\cdot\frac{\partial x}{x} - \frac{A}{a}\cdot\frac{(b\pm cx)\partial x}{a+bx\pm cx^2}$ dabit per (242. 241. 245. 255. §.) sequentia integralia.

$$\int\frac{A\partial x}{x(a+bx+cx^2)} = C + \frac{A}{a}\,l\left(\frac{x}{\sqrt{(a+bx+cx^2)}}\right) - \frac{Ab}{a\sqrt{(4ac-b^2)}}\,\mathrm{Arc Tang}\frac{2cx+b}{\sqrt{(4ac-b^2)}}.$$

$$\int\frac{A\partial x}{x(a+bx-cx^2)} = C + l\frac{A}{a}\left(\frac{x}{\sqrt{(a+bx-cx^2)}}\right) - \frac{Ab}{2a\sqrt{(4ac+b^2)}}\,l\left(\frac{\sqrt{(4ac+b^2)}+2cx-b}{\sqrt{(4ac+b^2)}-2cx+b}\right).$$

Scho-

Scholion.

Quantitates imaginariae in adplicatione harum formularum (254. 255. 256. §.) ad casus particulares eodem prorsus modo, quo in (253. §.), poterunt vitari.

257. Problema.

Dato integrali $\int u\,\partial x$ *invenire integrale* $\int u v\,\partial x$ *pro quibuscunque functionibus* v, u *variabilis* x.

Solutio.

Multiplicetur datum integrale $\int u\,\partial x$ per alteram functionem v, et seorsim per ejus exponentem differentialem ∂v; quaeratur deinde integrale exponentis differentialis $\partial v \int u\,\partial x$: si enim istud subtrahatur a producto $v\int u\,\partial x$; erit residuum aequale quaesito integrali $\int u v\,\partial x$.

Demonstratio.

Sit $\int u\,\partial x = z$; erit $v\int u\,\partial x = vz$, et $\partial v\int u\,\partial x = z\,\partial v$. Est autem $\partial . vz = v\,\partial z + z\,\partial v$: igitur $vz = \int v\,\partial z + \int z\,\partial v$, nimirum $v\int u\,\partial x = \int u v\,\partial x + \int \partial v \int u\,\partial x$; adeoque $\int u v\,\partial x = v\int u\,\partial x - \int \partial v \int u\,\partial x$.

258. Corollarium 1.

Pro trinomio $X = \alpha + \beta x^r + \gamma x^q$ et quibuscunque exponentibus r, q, m, p, posito $v = X^{p+1}$, et $u = x^{m-1}$, fiet (257. §.) $\int x^{m-1}\partial x X^{p+1} = X^{p+1}\int x^{m-1}\partial x - \int \partial . X^{p+1}\int x^{m-1}\partial x = \frac{X^{p+1}.x^m}{m} - \frac{(p+1)}{m}\int X^p x^m \partial X$: quare, cum sit $X = \alpha + \beta x^r + \gamma x^q$, et $\partial X = r\beta x^{r-1}\partial x + q\gamma x^{q-1}\partial x$, facta substitutione obtinebimus sequentem memorabilem formulam.

$$\int x^{m-1}\partial x X^{p+1} = \frac{x^m X^{p+1}}{m} - \frac{r\beta(p+1)}{m}\int x^{m+r-1}.\,\partial x X^p \\ - \frac{q\gamma(p+1)}{m}\int x^{m+q-1}.\,\partial x X^p.$$

259. Corollarium 2.

Ex hac aequatione pro $\int x^{m-1}\partial x X^{p+1} = \int x^{m-1}\partial x X^p(\alpha + \beta x^r + \gamma x^q) = \alpha\int x^{m-1}\partial x X^p + \beta\int x^{m+r-1}\partial x X^p + \gamma\int x^{m+q-1}\partial x X^p$ elicietur sequens formula.

$$\int x^{m+q-1}\partial x X^p = \frac{x^m X^{p+1}}{\gamma(qp+q+m)} - \frac{m\alpha}{\gamma(qp+q+m)}\int x^{m-1}\partial x X^p \\ - \frac{\beta(rp+r+m)}{\gamma(qp+q+m)}\int x^{m+r-1}\partial x X^p.$$

260. Co-

260. Corollarium 3.

Quodsi autem integrale (259. §.) substituatur in (258. §.), derivabitur inde sequens formula.

$$\int x^{m-1}\partial x X^{p+1} = \frac{x^m X^{p+1}}{qp+q+m} + \frac{\beta(q-r)(p+1)}{qp+q+m}\int x^{m+r-1}\partial x X^p + \frac{q\alpha(p+1)}{qp+q+m}\int x^{m-1}\partial x X^p.$$

261. Corollarium 4.

Hinc (259. 260. §.), posito $q=2$, $r=1$, hinc $X=\alpha+\beta x+\gamma x^2$, quod trinomium deinceps constanter litera X designabimus, fiunt sequentes formulae (I) (III): ex (I) porro nascitur (II), et ex (III) prodit (IV).

$$\text{I.}\int x^{m+1}\partial x X^p = \frac{x^m X^{p+1}}{\gamma(2p+2+m)} - \frac{m\alpha}{\gamma(2p+2+m)}\int x^{m-1}\partial x X^p - \frac{\beta(p+1+m)}{\gamma(2p+2+m)}\int x^m \partial x X^p,$$

$$\text{II.}\int x^{m-1}\partial x X^p = \frac{x^m X^{p+1}}{m\alpha} - \frac{\beta(p+1+m)}{m\alpha}\int x^m \partial x X^p - \frac{\gamma(2p+2+m)}{m\alpha}\int x^{m+1}\partial x X^p$$

$$\text{III.}\int x^{m-1}\partial x X^{p+1} = \frac{x^m X^{p+1}}{2p+2+m} + \frac{\beta(p+1)}{2p+2+m}\int x^m \partial x X^p + \frac{2\alpha(p+1)}{2p+2+m}\int x^{m-1}\partial x X^p.$$

$$\text{IV.}\int x^m \partial x X^p = -\frac{x^m X^{p+1}}{\beta(p+1)} - \frac{2\alpha}{\beta}\int x^{m-1}\partial x X^p + \frac{2(p+1)+m}{\beta(p+1)}\int x^{m-1}\partial x X^{p+1}.$$

262. Corollarium 5.

Si in (261. §. I. Form.) ponas $m=0$, et $p=-q$, obtinebis sequens integrale (I): quodsi autem in (261. §. III. Form.) ponas $p=-q$, et $m=1$, determinabis $\int\frac{\partial x}{X^q}$ per $\int\frac{\partial x}{X^{q-1}}$ et $\int\frac{x\partial x}{X^q}$, unde, substituto jam invento valore pro $\int\frac{x\partial x}{X^q}$, prodibit sequens formula (II)

$$\text{I.}\int\frac{x\partial x}{X^q} = \frac{-1}{2\gamma(q-1)X^{q-1}} - \frac{\beta}{2\gamma}\int\frac{\partial x}{X^q}.$$

$$\text{II.}\int\frac{\partial x}{X^q} = \frac{\beta+2\gamma x}{(q-1)(4\alpha\gamma-\beta^2)X^{q-1}} + \frac{2\gamma(2q-3)}{(q-1)(4\alpha\gamma-\beta^2)}\int\frac{\partial x}{X^{q-1}}.$$

263. Problema.

Invento integrali $\int\frac{\partial x}{X}$ *per* (253. §.), *determinare integralia* $\int\frac{\partial x}{X^n}$; $\int\frac{x\partial x}{X^n}$; $\int\frac{(A+Bx)\partial x}{X^n}$ *pro quocunque dato numero integro et positivo* n.

Solutio.

Solutio.

In (262. §. II.) ponatur successive $q=2$, $q=3$, $q=4$, - - - $q=n$; determinabitur eo ipso integrale $\int\frac{\partial x}{X^n}$, ut sequitur in (I); per istud autem dabitur integrale $\int\frac{x\partial x}{X^n}$ ob (262. §. I.): per utrumque demum determinari poterit etiam $\int\frac{(A+Bx)\partial x}{X^n}=A\int\frac{\partial x}{X^n}+B\int\frac{x\partial x}{X^n}$, ut sequitur in (II), nimirum pro valore integralis posterioris in (262. §.), invento.

$$\text{I. }\int\frac{\partial x}{X^n}=\frac{\beta+2\gamma x}{(n-1)(4\alpha\gamma-\beta^2)X^{n-1}}+\frac{(2n-3)2\gamma(\beta+2\gamma x)}{(n-1)(n-2)(4\alpha\gamma-\beta^2)^2X^{n-2}}$$
$$+\frac{(2n-3)(2n-5)(2\gamma)^2(\beta+2\gamma x)}{(n-1)(n-2)(n-3)(4\alpha\gamma-\beta^2)^3X^{n-3}}+ - - - - - - - -$$
$$+ - - - - - - - - + - - - - - - - - -$$
$$+\frac{(2n-3)(2n-5) - - - 5.3(2\gamma)^{n-2}(\beta+2\gamma x)}{(n-1)(n-2) - - - - 3.2.1.(4\alpha\gamma-\beta^2)^{n-1}.X}$$
$$+\frac{(2n-3)(2n-5) - - - - 5.3\ 1.(2\gamma)^{n-1}}{(n-1)(n-2) - - - - 3.2.1.(4\alpha\gamma-\beta^2)^{n-1}}\int\frac{\partial x}{X}.$$

$$\text{II. }\int\frac{(A+Bx)\partial x}{X^n}=\frac{-B}{2\gamma(n-1)X^{n-1}}+\frac{2\gamma A-\beta B}{2\gamma}\int\frac{\partial x}{X^n}.$$

264. Corollarium.

Pro $\beta=0$ obtinebimus ex (263. §. I.) sequens integrale, quod pro quovis numero integro positivo n pro perfecte determinato debet haberi, cum integrale $\int\frac{\partial x}{\alpha+\gamma x^2}$ per (252. §.) sit assignabile.

$$\int\frac{\partial x}{(\alpha+\gamma x^2)^n}=\frac{x}{(n-1)2\alpha(\alpha+\gamma x^2)^{n-1}}+\frac{(2n-3)x}{(n-1)(n-2)(2\alpha)^2(\alpha+\gamma x^2)^{n-2}}$$
$$+\frac{(2n-3)(2n-5)x}{(n-1)(n-2)(n-3)(2\alpha)^3(\alpha+\gamma x^2)^{n-3}}+ - - - - - - -$$
$$+ - - - - - - - - + - - - - - - - - -$$
$$+\frac{(2n-3)(2n-5) - - - - - 5.3.x}{(n-1)(n-2) - - - 3.2.1.(2\alpha)^{n-1}(\alpha+\gamma x^2)}$$
$$+\frac{(2n-3)(2n-5) - - - - 5.3.1.}{(n-1)(n-2) - - - - 3.2\ 1.(2\alpha)^{n-1}}\int\frac{\partial x}{\alpha+\gamma x^2}.$$

265. Problema.

Integrare exponentes differentiales $\frac{x^r \partial x}{X}$, $\frac{x^r \partial x}{X^n}$ *pro quibuslibet numeris integris et positivis* $r > 1$, $n > 1$.

Solutio.

Pro $-p$ et -1 loco p in (261. §. I. Form.) invenies sequentes formulas (I) (II): si vero in (261. §. III. Form.) scribas $-p$ loco p, elicies inde sequentem formulam (III). Ope secundae formulae poteris determinare integrale $\int \frac{x^r \partial x}{X}$, si quaeras integralia $\int \frac{\partial x}{X}$, $\int \frac{x \partial x}{X}$ per (253. 254. §.), deinde ponas in eadem formula successive $m = 1$, $m = 2$, $m = 3$, - - - $m = r - 1$. Ope primae autem formulae determinabitur integrale $\int \frac{x^r \partial x}{X^n}$, si quaeratur antea integrale $\int \frac{\partial x}{X^n}$ per (263. §.), deinde in sequente prima formula pro $p = n$ fiat successive $m = 0$, $m = 1$, $m = 2$, $m = 3$, $m = 4$, - - - $m = r - 1$. Haec tamen integratio ope primae formulae iis tantum casibus succedet, quibus fuerit $r < 2n - 1$; quamobrem, si sit $r = 2n - 1$, vel $r > 2n - 1$, conjungatur formula prima cum tertia, illaque integralia quaerantur ope formulae tertiae, quorum loco formula prima infiniti valoris quantitates exhibet, auxiliaria vero integralia, quibus in tertiae formulae adplicatione opus fuerit, determinentur ope primae.

$$\text{I.} \int \frac{x^{m+1} \partial x}{X^p} = \frac{x^m}{\gamma(m+2-2p)X^{p-1}} - \frac{m\alpha}{\gamma(m+2-2p)} \int \frac{x^{m-1} \partial x}{X^p} - \frac{\beta(m+1-p)}{\gamma(m+2-2p)} \int \frac{x^m \partial x}{X^p}.$$

$$\text{II.} \int \frac{x^{m+1} \partial x}{X} = \frac{x^m}{m\gamma} - \frac{\alpha}{\gamma} \int \frac{x^{m-1} \partial x}{X} - \frac{\beta}{\gamma} \int \frac{x^m \partial x}{X}.$$

$$\text{III.} \int \frac{x^m \partial x}{X^p} = \frac{x^m}{\beta(p-1)X^{p-1}} - \frac{2\alpha}{\beta} \int \frac{x^{m-1} \partial x}{X^p} - \frac{(m+2-2p)}{\beta(p-1)} \int \frac{x^{m-1} \partial x}{X^{p-1}}.$$

266. Corollarium 1.

Si in (265. §. II. Formul.) ponas $\beta = 0$, deinde successive $m = 2$, $m = 4$, $m = 6$, - - - $m = 2r$; obtinebis inde, ob $\int \frac{x \partial x}{\alpha + \gamma x^2} = \frac{1}{2\gamma} l(\alpha + \gamma x^2)$ per (245. §) sequens integrale, pro quovis numero positivo impari $2r + 1$ perfecte determinatum.

$\int \frac{x^{2r+1} \partial x}{\alpha + \gamma x^2}$

$$\int\frac{x^{2r+1}\partial x}{\alpha+\gamma x^2}=\frac{x^{2r}}{2r.\gamma}-\frac{\alpha x^{2r-2}}{(2r-2)\gamma^2}+\frac{\alpha^2 x^{2r-4}}{(2r-4)\gamma^3}-\frac{\alpha^3 x^{2r-6}}{(2r-6)\gamma^4}$$
$$+ - - - \pm\frac{\alpha^{r-1}.x^2}{2.\gamma^r}\mp\frac{\alpha^r}{2.\gamma^{r+1}}\log(\alpha+\gamma x^2).$$

267. Corollarium 2.

Si autem in (265. §. II. Form.) ponas $\beta=0$, tum successive $m=1$, $m=3$, $m=5$, - - - $m=2r-1$; invenies sequens integrale pro quolibet numero positivo pari $2r$ ope integralis per (252. §.) assignabilis perfecte determinatum.

$$\int\frac{x^{2r}\partial x}{\alpha+\gamma x^2}=\frac{x^{2r-1}}{(2r-1)\gamma}-\frac{\alpha x^{2r-3}}{(2r-3)\gamma^2}+\frac{\alpha^2 x^{2r-5}}{(2r-5)\gamma^3}-\frac{\alpha^3 x^{2r-7}}{(2r-7)\gamma^4}$$
$$+ - - - - - \pm\frac{\alpha^{r-1}.x}{1.\gamma^r}\mp\frac{\alpha^r}{\gamma^r}\int\frac{\partial x}{\alpha+\gamma x^2}.$$

268. Corollarium 3.

Si aequationem (III) in (265. §.) multiplices per β, tum ponas $\beta=0$, et $m+1$ loco m; elicies inde integrale $\int\frac{x^m\partial x}{(\alpha+\gamma x^2)^p}$ determinatum per $\int\frac{x^m\partial x}{(\alpha+\gamma x^2)^{p-1}}$: quodsi ergo in hac formula successive ponas $p=2$, $p=3$, $p=4$, - - - $p=n$, obtinebis, ut sequitur, integrale $\int\frac{x^m\partial x}{(\alpha+\gamma x^2)^n}$ pro quibuslibet numeris integris positivis m, n perfecte determinatum, ob (266. 267. §.).

$$\int\frac{x^m\partial x}{(\alpha+\gamma x^2)^n}=\frac{x^{m+1}}{(n-1)2\alpha(\alpha+\gamma x^2)^{n-1}}-\frac{(m-(2n-3))x^{m+1}}{(n-1)(n-2)(2\alpha)^2(\alpha+\gamma x^2)^{n-2}}$$
$$+\frac{(m-(2n-3))(m-(2n-5))x^{m+1}}{(n-1)(n-2)(n-3)(2\alpha)^3(\alpha+\gamma x^2)^{n-3}} - - - - - -$$
$$+ - - - - - - - - - - - - - - - - - -$$
$$+\frac{(m-(2n-3))(m-(2n-5))\ - - -\ (m-3)x^{m+1}}{(n-1)(n-2)\ - - - -\ 3.2.1.(2\alpha)^{n-1}(\alpha+\gamma x^2)}$$
$$\mp\frac{(m-(2n-3))(m-(2n-5))\ - - -\ (m-1)}{(n-1)(n-2)\ - - - -\ 3.2.1.(2\alpha)^{n-1}}\int\frac{x^m\partial x}{\alpha+\gamma x^2}.$$

269. Problema.

Integrare exponentes differentiales $\frac{\partial x}{xX^n}$, $\frac{\partial x}{x^rX^n}$ *pro quibuslibet numeris integris positivis* r, n.

Solutio.

Solutio.

Pro $-m$ loco m in (265. §. I.) invenietur sequens formula (I): si autem in (261. §. III.) ponatur $-p$ loco p, et $m=0$; reperietur sequens formula (II). Iam vero ope secundae formulae determinabitur integrale $\int\frac{\partial x}{xX^n}$, si in illa ponatur successive $p=2, p=3, p=4, \text{----} p=n$, integraliaque auxiliaria $\int\frac{\partial x}{xX}, \int\frac{\partial x}{X^2}, \int\frac{\partial x}{X^3}$, etc. per (256. 263. §.) quaerantur. Et ope primae formulae determinabitur integrale $\int\frac{\partial x}{x^r X^n}$, si ope secundae quaeratur $\int\frac{\partial x}{xX^n}$, hacque ipsa operatione definiatur quoque $\int\frac{\partial x}{X^n}$, deinde in formula prima pro $p=n$ fiat successive $m=1, m=2, m=3, m=4, \text{---} m=r-1$.

$$\text{I. } \int\frac{\partial x}{x^{m+1}X^p}=\frac{-1}{m\alpha x^m X^{p-1}}-\frac{\beta(p+m-1)}{m\alpha}\int\frac{\partial x}{x^m X^p}-\frac{\gamma(2p+m-2)}{m\alpha}\int\frac{\partial x}{x^{m-1}X^p}.$$

$$\text{II. } \int\frac{\partial x}{xX^p}=\frac{1}{2\alpha(p-1)X^{p-1}}-\frac{\beta}{2\alpha}\int\frac{\partial x}{X^p}+\frac{1}{\alpha}\int\frac{\partial x}{xX^{p-1}}.$$

270. Corollarium 1.

Posito in (269. §. I.) $\beta=0$, $p=1$, tum successive $m=1, m=3, m=5, \text{---} m=2r-1$; obtinebitur sequens integrale pro quovis numero positivo pari $2r$ per integrale juxta (252. §.) assignabile perfecte determinatum.

$$\int\frac{\partial x}{x^{2r}(\alpha+\gamma x^2)}=\frac{-1}{(2r-1)\alpha x^{2r-1}}+\frac{\gamma}{(2r-3)\alpha^2 x^{2r-3}}-\frac{\gamma^2}{(2r-5)\alpha^3 x^{2r-5}}$$
$$+\frac{\gamma^3}{(2r-7)\alpha^4 x^{2r-7}}-\text{ - - - }+\text{ - - - -}$$
$$\pm\frac{\gamma^{r-1}}{1\cdot\alpha^r\cdot x}\pm\frac{\gamma^r}{\alpha^r}\int\frac{\partial x}{\alpha+\gamma x^2}+C.$$

271. Corollarium 2.

Cum pro $b=0$, $a=\alpha$, $\pm c=\gamma$ in (256. §.) prodeat $\int\frac{\partial x}{x(\alpha+\gamma x^2)} = \frac{1}{\alpha}\log\left(\frac{x}{\sqrt{(\alpha+\gamma x^2)}}\right)$, si in (269. §. I.) ponatur $\beta=0$, $p=1$, tum

tum successive $m=2$, $m=4$, $m=6$, $m=8$ - - - $m=2r$; determinabitur sequens integrale pro quovis numero positivo impari $2r+1$.

$$\int\frac{\partial x}{x^{2r+1}(\alpha+\gamma x^2)}=\frac{-1}{2r\,.\,\alpha x^{2r}}+\frac{\gamma}{(2r-2)\alpha^2 x^{2r-2}}-\frac{\gamma^2}{(2r-4)\alpha^3 x^{2r-4}}$$
$$+\frac{\gamma^3}{(2r-6)\alpha^4 x^{2r-6}}-\;-\;-\;-\;+\;-\;-\;-\;-$$
$$\pm\frac{\gamma^{r-1}}{2\alpha^r\,.\,x^2}\pm\frac{\gamma^r}{\alpha^{r+1}}\log\left(\frac{x}{\sqrt{(\alpha+\gamma x^2)}}\right)+C.$$

272. Corollarium 3.

Cognito autem integrali $\int\frac{\partial x}{x^m(\alpha+\gamma x^2)}$ pro quocunque numero integro positivo m, pari aut impari, (270. 271. §.); determinabitur integrale $\int\frac{\partial x}{x^m(\alpha+\gamma x^2)^n}$ pro quovis alio numero integro positivo n, si $-m$ loco m ponatur in (268. §.): erit enim

$$\int\frac{\partial x}{x^m(\alpha+x\gamma^2)^n}=\frac{1}{(n-1)2\alpha x^{m-1}(\alpha+\gamma x^2)^{n-1}}+\frac{m+(2n-3)}{(n-1)(n-2)(2\alpha)^2 x^{m-1}(\alpha+\gamma x^2)^{n-2}}$$
$$+\frac{(m+(2n-3))(m+(2n-5))}{(n-1)(n-2)(n-3)(2\alpha)^3 x^{m-1}(\alpha+\gamma x^2)^{n-3}}+\;-\;-\;-\;-\;-$$
$$+\;-\;-\;-\;-\;-\;-\;-\;-\;-\;+\;-\;-\;-\;-\;-\;-\;-\;-\;-$$
$$+\frac{(m+(2n-3))(m+(2n-5))\;-\;-\;-\;-\;(m+3)}{(n-1)(n-2)\;-\;-\;-\;-\;3.2.1.(2\alpha)^{n-1}x^{m-1}(\alpha+\gamma x^2)}$$
$$+\frac{(m+(2n-3))(m+(2n-5))\;-\;-\;-\;-\;(m+1)}{(n-1)(n-2)\;-\;-\;-\;-\;-\;-\;3.2.1.(2\alpha)^{n-1}}\int\frac{\partial x}{x^m(\alpha+\gamma x^2)}.$$

273. Problema.

Integrare exponentes differentiales formula generali $\partial y=\frac{x^k\partial x}{(\alpha+\beta x^e+\gamma x^{2e})^p}$ *comprehensos, quidquid sint exponentes* k, e, *modo sit* p *numerus integer negativus, et* $\frac{k+1}{e}$ *numerus integer, positivus aut negativus.*

Solutio.

Fiat $z=x^e$; erit $x=z^{\frac{1}{e}}$; $x^k=z^{\frac{k}{e}}$; $\partial x=\frac{1}{e}z^{\frac{1}{e}-1}\partial z$:

adeoque habebimus

$$\partial y = \frac{x^k \partial x}{(\alpha+\beta x^e+\gamma x^{2e})^p} = \frac{1}{e} \cdot \frac{z^{\frac{k+1}{e}-1} \cdot \partial z}{(\alpha+\beta z+\gamma z^2)^p}.$$

Cum igitur sit p numerus integer positivus, et $\frac{k+1}{e}$ numerus integer positivus vel negativus; poterit integratio per (265. 269. §.) omni casu perfici.

Scholion.

Secundum haec principia commodissime perficietur integratio omnium exponentium differentialium fractorum et rationalium formula generali

$$\partial y = \frac{(A+Bx^a+Cx^b+Dx^c+ \,-\,-\,-\, +Qx^p)\partial x}{x^s X^r}$$

comprehensorum, quidquid sint numeri integri a, b, c, - - - p, s, r, modo X aut sit simplex, aut quadratica functio, puta $X=\alpha+\beta x$, aut $X=\alpha+\gamma x^2$, vel $X=\alpha+\beta x+\gamma x^2$. Cum enim debeat esse

$$\partial y = \frac{A\partial x}{x^s X^r} + \frac{Bx^{a-s}\partial x}{X^r} + \frac{Cx^{b-s}\partial x}{X^r} + \,-\,-\,-\, + \frac{Qx^{p-s}\partial x}{X^r},$$

singulaeque partes hujus exponentis differentialis, pro numeris integris s, a—s, b—s, - - - p—s, tam positivis quam negativis, ope praecedentium formularum facile possint integrari; poterit eo ipso etiam integrale y ope earundem formularum perfecte determinari (242. §.). Inter omnes alios exponentes rationales fractos maxime memorabiles sunt, qui formula generali $\frac{x^m \partial x}{(a+bx^n)^p}$ pro $n>2$ comprehendentur: integralia ipsorum perfecte possunt determinari, eadem fere ratione, qua superius integralia ejusmodi exponentium differentialium pro $n=2$ determinavimus (264. 266. 267. 268. 270. 271. 272. §.) Prius tamen, quam haec exponantur, methodum generalem integrandi omnes exponentes differentiales fractos et rationales paucis attingemus.

274. Problema.

Datis factoribus simplicibus formae $p+qx$, *et quadraticis formae* $\alpha+\beta x+\gamma x^2$, *in quos functio integra rationalis* N *sit resolubilis, invenire integrale exponentis differentialis fracti et rationalis* $\frac{M\partial x}{N}$.

Solutio.

Solutio.

Si $\frac{M\varepsilon x}{N}$ esset fractio impropria, posset ea resolvi in duas partes $F\varepsilon x$, $\frac{m\varepsilon x}{N}$, nimirum functionem integram $F\varepsilon x$ et aliquam fractionem propriam $\frac{m\varepsilon x}{N}$ (10. 11. §.), essetque $\int\frac{M\varepsilon x}{N}=\int F\varepsilon x+\int\frac{m\varepsilon x}{N}$ (242. §.): quare, cum integrale $\int F\varepsilon x$ pro quavis possibili functione integra et rationali F variabilis x per (250. 251.) facile determinetur, patet, totum negotium eo demum reduci, ut ostendatur, quo modo integratio cujuslibet fractionis genuinae $\frac{m\varepsilon x}{N}$, qualem idcirco $\frac{M\varepsilon x}{N}$ esse supponemus, possit perfici.

Cum, per hypothesin, noscantur omnes factores simplices, aut quadratici, vel simplices et quadratici, in quos denominator N est resolubilis; resolvatur exponens differentialis $\frac{M\varepsilon x}{N}$ in fractiones partiales $\frac{A\varepsilon x}{p+qx}$, $\frac{(A+Bx)\varepsilon x}{\alpha+\beta x+\gamma x^2}$; $\frac{A\varepsilon x}{(p+qx)^n}$, $\frac{(A+Bx)\varepsilon x}{(\alpha+\beta x+\gamma x^2)^n}$, quarum summa aequalis sit eidem exponenti $\frac{M\varepsilon x}{N}$, quod per praecepta 3*tii* capitis omni casu poterit praestari: deinde integrentur singulae fractiones seorsim secundum principia superius stabilita; summa enim integralium omnium fractionum dabit integrale exponentis differentialis $\frac{M\varepsilon x}{N}$.

275. Problema.

Sit functio $a+bx^n$ *integra rationalis composita ex factore simplici* $e-fx$, *quadratico* $d^2-c^2x^2$, *et pluribus aliis factoribus quadraticis, qui omnes formulâ generali* $p^2-2pqx\,\mathrm{Cos}\,\varphi+q^2x^2$ *comprehendantur, ita ut singulos inde ordine derivare liceat, si loco arcus indeterminati* φ *certi arcus* k, l, - - - - - s *successive substituantur: exhibere integrale perfectum exponentis rationalis* $\frac{x^m\varepsilon x}{a+bx^n}$.

Solutio.

1. Una fractionum partialium, in quas exponens $\frac{x^m\varepsilon x}{a+bx^n}$ potest resolvi, sequitur in (I) ob (226. §.); altera in (II) ob (228. §.); omnes vero

reliquae

reliquae, ob (234. §.), comprehenduntur in (III), ita ut omnes inde ordine sint proditurae, si loco arcus indeterminati φ certi arcus $k, l, \ldots s$ successive substituantur.

$$\text{I. } \frac{-f^{n-m}.\,\partial x}{nbe^{n-m-1}(e-fx)} = \frac{f^{n-m-1}}{nbe^{n-m-1}}\cdot\frac{-f\partial x}{e-fx}.$$

$$\text{II. } \frac{-2c^{n-m}.\,\partial x}{nbd^{n-m-2}(d^2-c^2x^2)},\ \text{aut}\ \frac{c^{n-m-1}}{nbd^{n-m-1}}\cdot\frac{-2c^2x\partial x}{d^2-c^2x^2},$$

prout est $n-m$ par aut impar numerus.

$$\text{III. } \frac{2p^{m+1}}{naq^m}\cdot\frac{(p\,\mathrm{Cos}\,m\varphi - qx\,\mathrm{Cos}(m+1)\varphi)\partial x}{p^2-2pqx\,\mathrm{Cos}\,\varphi+q^2x^2}.$$

2. Iam vero integrale exponentis differentialis (I) sequitur in (IV) per (241. 245. §.): integrale autem exponentis (II) sequitur in (V) per (241. 245. 252. §.). Porro sit

$$a=p^2;\ b=-2pq\,\mathrm{Cos}\,\varphi;\ c=q^2;$$
$$A=p\,\mathrm{Cos}\,m\varphi;\ B=-q\,\mathrm{Cos}(m+1)\varphi:$$

erit $2Ac-Bb=2pq^2(\mathrm{Cos}\,m\varphi-\mathrm{Cos}(m+1)\varphi.\mathrm{Cos}\,\varphi)=$
$=2pq^2(\mathrm{Cos}\,m\varphi-(\mathrm{Cos}\,m\varphi-\mathrm{Sin}(m+1)\varphi.\mathrm{Sin}\,\varphi);$

hinc $2Ac-Bb=2pq^2\,\mathrm{Sin}(m+1)\varphi.\mathrm{Sin}\,\varphi.$

$\sqrt{(4ac-b^2)}=2pq\sqrt{(1-\mathrm{Cos}\,\varphi^2)}=2pq\,\mathrm{Sin}\,\varphi.$

Quare, si hos valores in (255. §.) substituas, tum totum in $\frac{2p^{m+1}}{naq^m}$ duces, obtinebis integrale exponentis differentialis (III), ut sequitur in (VI).

$$\text{IV. } \frac{f^{n-m-1}}{nbe^{n-m-1}}\log(e-fx).$$

$$\text{V. } \frac{c^{n-m-1}}{nbd^{n-m-1}}\log\frac{d-cx}{d+cx},\ \text{aut}\ \frac{c^{n-m-1}}{nbd^{n-m-1}}\log(d^2-c^2x^2),$$

consequenter

$$\frac{c^{n-m-1}}{nbd^{n-m-1}}\Big(\log(d-cx)\mp l(d+cx)\Big)\ \text{prout est}\ n-m\ \begin{matrix}\text{num. par}\\ \text{vel impar.}\end{matrix}$$

$$\text{VI. } \frac{p^{m+1}}{naq^{m+1}}\Big(2\,\mathrm{Sin}(m+1)\varphi.\ \mathrm{Arc\ Tang}\,\frac{qx-p\,\mathrm{Cos}\,\varphi}{p\,\mathrm{Sin}\,\varphi}$$
$$-\mathrm{Cos}(m+1)\varphi.\ l(p^2-2pqx\,\mathrm{Cos}\,\varphi+q^2x^2)\Big).$$

3. Ob.

3. Ob (1) constabit ergo integrale dati exponentis differentialis $\frac{x^m \partial x}{a+bx^n}$ ex (IV), (V), et omnibus integralibus, quae ex (VI) possunt derivari, si loco φ ii determinati arcus k, l, - - - s successive substituantur, pro quibus loco φ substitutis ex factore $p^2 - 2pq\,x\,\text{Cos}\,\varphi + q^2 x^2$ omnes factores quadratici functionis $a+bx^n$ vi propositi problematis eliciuntur.

276. Corollarium 1.

Functio integra rationalis $g+x^n$ composita est ex factoribus quadraticis, qui ex $g^{\frac{2}{n}} - 2g^{\frac{1}{n}}\,x\,\text{Cos}\,\frac{(2k+1)}{n}\pi + x^2$ ordine elicientur, si successive fiat $k=0, k=1, k=2, k=3$, et sic porro usque $k=\frac{1}{2}n-1$, vel $k=\frac{n-3}{2}$ inclusive, prout est n par vel impar numerus, ita tamen, ut hoc altero casu illa functio adhuc unum factorem simplicem $g^{\frac{1}{n}}+x$ habeat (232. §.): quodsi ergo in (275. §. IV. VI.) ponas $e=g^{\frac{1}{n}}$, $f=-1$, $b=1$, $a=g$, $p=g^{\frac{1}{n}}$, $q=1$, $\varphi=\frac{(2k+1)}{n}\pi$, obtinebis sequentes formulas, per quas integrale exponentis differentialis $\frac{x^m \partial x}{g+x^n}$ perfecte determinatum est: nimirum obtinebis illud pro numero pari n ex formula (II), si loco k terminos seriei 0, 1, 2, 3 - - - - $\frac{1}{2}n-1$ successive substituas; pro numero autem impari n sumes integrale (I), et huic addes integralia, quae successiva substitutione terminorum seriei 0, 1, 2, 3, 4 - - - - $\frac{n-3}{2}$ loco k ex (II) possunt elici.

Adhibito

I. $\dfrac{\pm \log(g^{\frac{1}{n}}+x)}{ng^{\frac{n-m-1}{n}}}$ signo + pro pari $n-m-1$, et − pro impari $n-m-1$.

$$\text{II.}\ \left(2\,\text{Sin}\,\frac{(m+1)(2k+1)}{n}\pi.\,\text{Arc Tang}\,\frac{x-g^{\frac{1}{n}}\,\text{Cos}\,\frac{(2k+1)}{n}\pi}{g^{\frac{1}{n}}\,\text{Sin}\,\frac{(2k+1)}{n}\pi}\right.$$

$$\left.-\text{Cos}\,\frac{(m+1)(2k+1)}{n}\pi.\,l\left(g^{\frac{2}{n}}-2g^{\frac{1}{n}}\,x\,\text{Cos}\,\frac{(2k+1)}{n}\pi+x^2\right)\right).\ \frac{1}{ng^{\frac{n-m-1}{n}}}.$$

277. Corollarium 2.

Posito $m=0$ in (276. §.) obtinebis sequentes formulas, integrale exponentis differentialis $\frac{\partial x}{g+x^n}$ pro quolibet numero integro positivo, pari, et impari n ea lege, quam in (276. §.) determinavimus, exhibentes.

Adhibito

I. $\dfrac{\pm \log (g^{\frac{1}{n}}+x)}{ng^{\frac{n-1}{n}}}$ signo + pro pari $n-1$, et − pro impari $n-1$.

$$\text{II.}\ \left(2\,\mathrm{Sin}\frac{(2k+1)}{n}\pi.\,\mathrm{ArcTang}\frac{x-g^{\frac{1}{n}}\,\mathrm{Cos}\frac{(2k+1)}{n}\pi}{g^{\frac{1}{n}}\,\mathrm{Sin}\frac{(2k+1)}{n}\pi}\right.$$

$$\left.-\mathrm{Cos}\frac{(2k+1)}{n}\pi.\,l\left(g^{\frac{2}{n}}-2g^{\frac{1}{n}}x\,\mathrm{Cos}\frac{(2k+1)}{n}\pi+x^2\right)\right).\ \frac{1}{ng^{\frac{n-1}{n}}}.$$

278. Corollarium 3.

Quodsi autem ponas $-m$ loco m in (276. §.), prodibunt inde sequentes formulae, integrale perfectum exponentis differentialis $\frac{\partial x}{x^m(g+x^n)}$ pro quibuslibet numeris integris et positivis m, n ea lege, quam in (276. §.) exposuimus, exhibentes.

Adhibito

I. $\dfrac{\pm \log (g^{\frac{1}{n}}+x)}{ng^{\frac{n+m-1}{n}}}$ signo + pro pari $n+m-1$, et − pro impari $n+m-1$.

$$\text{II.}\ \left(-2\,\mathrm{Sin}\frac{(m-1)(2k+1)}{n}\pi.\ \mathrm{ArcTang}\frac{x-g^{\frac{1}{n}}\,\mathrm{Cos}\frac{(2k+1)}{n}\pi}{g^{\frac{1}{n}}\,\mathrm{Sin}\frac{(2k+1)}{n}\pi}\right.$$

$$\left.+\,\mathrm{Cos}\frac{(m-1)(2k+1)}{n}\pi.\,l\left(g^{\frac{2}{n}}-2g^{1}x\,\mathrm{Cos}\frac{(2k+1)}{n}\pi+x^2\right)\right).\ \frac{1}{ng^{\frac{n+m-1}{n}}}$$

279. Corollarium 4.

Functio integra et rationalis $g-x^n$ sic est composita ex factoribus, ut expressio $g^{\frac{2}{n}}-2g^{\frac{1}{n}}x\,\mathrm{Cos}\frac{2k}{n}\pi+x^2$ singulos ejus factores quadraticos dare

dare debeat, si in illa successive fiat $k=1$, $k=2$, $k=3$, et sic porro usque $k=\frac{1}{2}n-1$, vel $k=\frac{n-1}{2}$, prout est n par vel impar numerus, ita tamen, ut ad illos praeterea factor $g^{\frac{2}{n}}-x^2$ casu numeri paris n, vel factor simplex $g^{\frac{1}{n}}-x$ casu numeri imparis n debeat accedere (233. §.): quare si in (275. §. IV. V. VI.) ponas $e=d=p=g^{\frac{1}{n}}$, $q=c=f$, $=1$, $a=g$, $b=-1$, $\varphi=\frac{2k}{n}\pi$, prodibunt inde sequentes formulae, quae integrale exponentis differentialis $\frac{x^m\,\partial x}{g-x^n}$ pro quovis alio numero integro m hac lege complectuntur. Si n est numerus par; sumatur integrale (II), eique addantur omnia integralia, quae successiva substitutione terminorum seriei 1, 2, 3, 4, - - - $\frac{1}{2}n-1$ loco k ex (III) possunt derivari: si vero n est numerus impar; sumendum est integrale (I), cui addi debent omnia integralia, quae successiva substitutione terminorum seriei 1, 2, 3, 4, - - - $\frac{n-1}{2}$ loco k ex (III) possunt elici.

$$\text{I. } \frac{-\log\left(g^{\frac{1}{n}}-x\right)}{ng^{\frac{n-m-1}{n}}}.$$

$$\text{II. } \frac{-l\left(g^{\frac{1}{n}}-x\right)\pm l\left(g^{\frac{1}{n}}+x\right)}{ng^{\frac{n-m-1}{n}}}.$$

Adhibito signo + pro pari $n-m$, et − pro impari $n-m$.

$$\text{III. } \left(2\,\mathrm{Sin}\frac{(m+1)2k}{n}\pi.\ \mathrm{Arc\ Tang}\frac{x-g^{\frac{1}{n}}\mathrm{Cos}\frac{2k}{n}\pi}{g^{\frac{1}{n}}\mathrm{Sin}\frac{2k}{n}\pi}\right.$$

$$\left.-\mathrm{Cos}\frac{(m+1)2k}{n}\pi.\ l\left(g^{\frac{2}{n}}-2g^{\frac{1}{n}}x\,\mathrm{Cos}\frac{2k}{n}\pi+x^2\right)\right).\ \frac{1}{ng^{\frac{n-m-1}{n}}}.$$

280. Corollarium 5.

Pro $m=0$ prodibunt hinc sequentes formulae, integrale exponentis differentialis $\frac{\partial x}{g-x^n}$ eadem prorsus lege (279. §.) complectentes.

I. $\dfrac{-\log(g^{\frac{1}{n}}-x)}{ng^{\frac{n-1}{n}}}$.

Adhibito

II. $\dfrac{-l(g^{\frac{1}{n}}-x)\pm(g^{\frac{1}{n}}+x)}{ng^{\frac{n-1}{n}}}$ figno + pro pari n, et — pro impari n.

III. $$\Big(2\,\mathrm{Sin}\frac{2k}{n}\pi.\ \mathrm{Arc\ Tang}\frac{x-g^{\frac{1}{n}}\,\mathrm{Cof}\frac{2k}{n}\pi}{g^{\frac{1}{n}}\,\mathrm{Sin}\frac{2k}{n}\pi}$$

$$-\mathrm{Cof}\frac{2k}{n}\pi.\,l\Big(g^{\frac{2}{n}}-2g^{\frac{1}{n}}x\,\mathrm{Cof}\frac{2k}{n}\pi+x^2\Big)\Big).\ \frac{1}{ng^{\frac{n-1}{n}}}$$

281. Corollarium 6.

Et pro —m loco m obtinebimus ex (279. §.) fequentes formulas, quae eadem lege (279. §.) exhibent perfectum integrale exponentis differentialis $\frac{\partial x}{x^m(g-x^n)}$ pro quibuslibet numeris integris et pofitivis m, n.

I. $\dfrac{-\log(g^{\frac{1}{n}}-x)}{ng^{\frac{n+m-1}{n}}}$.

Adhibito

II. $\dfrac{-l(g^{\frac{1}{n}}-x)\pm l(g^{\frac{1}{n}}+x)}{ng^{\frac{n+m-1}{n}}}$ figno + pro pari n+m, et — pro impari n+m.

III. $$\Big(-2\,\mathrm{Sin}\frac{(m-1)2k}{n}\pi.\ \mathrm{Arc\ Tang}\frac{x-g^{\frac{1}{n}}\,\mathrm{Cof}\frac{2k}{n}\pi}{g^{\frac{1}{n}}\,\mathrm{Sin}\frac{2k}{n}\pi}$$

$$+\mathrm{Cof}\frac{(m-1)2k}{n}\pi.\,l\Big(g^{\frac{2}{n}}-2g^{\frac{1}{n}}.x\,\mathrm{Cof}\frac{2k}{n}\pi+x^2\Big)\Big)\frac{1}{ng^{\frac{n+m-1}{n}}}.$$

282. Problema.

Integrare exponentem differentialem $x^m\partial x(a+bx^n)^p$ *pro quibuscunque exponentibus* m, n, p, *integris et fractis, modo fit* $\frac{m+1}{n}$ *numerus integer pofitivus.*

Solutio.

Solutio.

Sit $z = a + bx^n$; erit $x = \frac{(z-a)^{\frac{1}{n}}}{b^{\frac{1}{n}}}$; $\partial x = \frac{\partial z\,(z-a)^{\frac{1-n}{n}}}{nb^{\frac{1}{n}}}$:

pro his valoribus, posito $\frac{m+1}{n} = \varphi$, fiet

$$x^m \partial x\,(a+bx^n)^p = \frac{z^p \partial z\,(z-a)^{\frac{m+1}{n}-1}}{nb^{\frac{m+1}{n}}} = \frac{z^p \partial z\,(z-a)^{\varphi-1}}{nb^{\varphi}}.$$

Explicetur jam $(z-a)^{\varphi-1}$ per seriem (52. §.), deinde, ductis omnibus terminis in $z^p \partial z$, quaeratur integrale per (249. §.), ac demum multiplicetur id per $\frac{1}{nb^{\varphi}}$ (241. §.), restituaturque valor variabilis z; obtinebitur pro quaesito integrali sequens expressio.

$$\varphi = \frac{m+1}{n}.$$

$$\begin{aligned}\int x^m \partial x\,(a+bx^n)^p = \frac{1}{nb^{\varphi}}\Bigg(&\frac{(a+bx^n)^{p+\varphi}}{p+\varphi} - \frac{(\varphi-1)a}{1}\cdot\frac{(a+bx^n)^{p+\varphi-1}}{p+\varphi-1}\\ &+\frac{(\varphi-1)(\varphi-2)a^2}{1\cdot 2}\cdot\frac{(a+bx^n)^{p+\varphi-2}}{p+\varphi-2}\\ &-\frac{(\varphi-1)(\varphi-2)(\varphi-3)a^3}{1\cdot 2\cdot 3}\cdot\frac{(a+bx^n)^{p+\varphi-3}}{p+\varphi-3}\\ &+ - - - - - - - - - - -\\ &\pm\frac{(\varphi-1)(\varphi-2)(\varphi-3)\;-\;-\;-\;(\varphi-r)a^r}{1\cdot 2\cdot 3\;-\;-\;-\;-\;-\;-\;-\;r}\cdot\frac{(a+bx^n)^{p+\varphi-r}}{p+\varphi-r}\Bigg)+C.\end{aligned}$$

Ultimus terminus est indeterminatus indicis r, ex quo singuli termini, post primum ordine sequentes, eliciuntur, si successive fiat $r=1$, $r=2$, $r=3$, $r=4$; et sic porro. Unde perspicuum fit, quod, cum $\varphi = \frac{m+1}{n}$ per hypothesin sit numerus integer positivus, ubi ad $r=\varphi$ deventum fuerit, terminus ultimus, ob $\varphi - r = 0$, futurus sit aequalis nihilo: integrale determinabitur ergo per terminos numero $\varphi = \frac{m+1}{n}$. Porro fieri potest, ut, priusquam abrumpatur series, evadat $r = \varphi + p$: hoc casu erit

$p+\varphi-r=0$, et terminus generalis dabit $\frac{(a+bx^n)^0}{0}$, cujus loco sumi debebit $\log(a+bx^n)$ ob (246. §.).

283. Problema.

Integrare exponentem differentialem $x^m \partial x (a+bx^n)^p$ *pro quibuscunque numeris* m, n, p, *modo sit* $\frac{m+1}{n}+p$ *numerus integer negativus.*

Solutio.

Cum sit $x^m \partial x (a+bx^n)^p = x^{m+np} . \partial x (ax^{-n}+b)^p$; ponatur in (282. §.) $m+np$ loco m, $-n$ loco n, a loco b, b loco a, et $ax^{-n}+b = \frac{a+bx^n}{x^n}$ loco $a+bx^n$ obtinebitur sequens series.

$$\varphi = -\left(\frac{m+1}{n}+p\right).$$

$$\int x^m \partial x (a+bx^n)^p = \frac{1}{na\varphi}\left(-\frac{\left(\frac{a+bx^n}{x^n}\right)^{p+\varphi}}{p+\varphi} + \frac{(\varphi-1)b}{1} . \frac{\left(\frac{a+bx^n}{x^n}\right)^{p+\varphi-1}}{p+\varphi-1}\right.$$

$$- \frac{(\varphi-1)(\varphi-2)b^2}{1\ .\ 2} . \frac{\left(\frac{a+bx^n}{x^n}\right)^{p+\varphi-2}}{p+\varphi-2}$$

$$+ \frac{(\varphi-1)(\varphi-2)(\varphi-3)b^3}{1\ .\ 2\ .\ 3} . \frac{\left(\frac{a+bx^n}{x^n}\right)^{p+\varphi-3}}{p+\varphi-3}$$

$$- - - - - - + - - - - - - - - - -$$

$$\left.\pm \frac{(\varphi-1)(\varphi-2) - - - (\varphi-r)b^r}{1\ .\ 2 - - - - - r} . \frac{\left(\frac{a+bx^n}{x^n}\right)^{p+\varphi-r}}{p+\varphi-r}\right) + C.$$

In hac quoque serie terminus ultimus indeterminatus est, qui singulos terminos, post primum sequentes, ordine dabit, si in illo fiat successive $r=1$, $r=2$, $r=3$, $r=4$, et ita porro. Cum vero, per hypothesin, $\frac{m+1}{n}+p$ sit numerus integer negativus, adeoque φ integer positivus; necesse est, ut, ubi fuerit $r=\varphi$, ob $\varphi-r=0$, terminus ultimus aequetur nihilo; constabit ergo integrale terminis numero $\varphi = -\left(\frac{m+1}{n}\right.$

Deinde eveniet fors, ut, antequam fiat $r=\varphi$, evadat $r=\varphi+p$, pro quo valore continebit terminus ultimus expressionem $\frac{\left(\frac{a+bx^n}{x^n}\right)^0}{0}$, quae denotabit $\log\left(\frac{a+bx^n}{x^n}\right)$ per (246. §.).

284. Problema.

Datis integralibus $\int\frac{x^m\partial x}{a+bx^n}$, $\int\frac{\partial x}{a+bx^n}$, $\int\frac{\partial x}{x^m(a+bx^n)}$, *pro quibuscunque numeris integris positivis* m, n; *invenire integralia* $\int\frac{x^m\partial x}{(a+bx^n)^p}$, $\int\frac{\partial x}{(a+bx^n)^p}$, $\int\frac{\partial x}{x^m(a+bx^n)^p}$ *pro quolibet numero integro et positivo* $p>1$.

Solutio.

Ponatur in (258. §.) $\alpha=a$, $\gamma=b$, $q=n$, $\beta=0$, hinc $X=a+bx^n$, deinde $p=-k$, et $m+1$ loco m, obtinebitur ex (260. §.) pro his valoribus sequens formula (I). Quodsi jam in hac formula successive fiat $k=2$, $k=3$, $k=4$, - - - $k=p$; prodibit inde integrale (II); et ex hoc sequuntur, pro $m=0$, et $-m$ loco m, integralia (III) (IV.).

$$\text{I.}\int\frac{x^m\partial x}{(a+bx^n)^k}=\frac{x^{m+1}}{na(k-1)(a+bx^n)^{k-1}}+\frac{n(k-1)-(m+1)}{na(k-1)}\int\frac{x^m\partial x}{(a+bx^n)^{k-1}}.$$

$$\begin{aligned}\text{II.}\int\frac{x^m\partial x}{(a+bx^n)^p}=\Bigg(&\frac{1}{(p-1)na(a+bx^n)^{p-1}}+\frac{n(p-1)-(m+1)}{(p-1)(p-2)(na)^2(a+bx^n)^{p-2}}\\&+\frac{(n(p-1)-(m+1))(n(p-2)-(m+1))}{(p-1)(p-2)(p-3)(na)^3(a+bx^n)^{p-3}}\\&+\frac{(n(p-1)-(m+1))(n(p-2)-(m+1))(n(p-3)-(m+1))}{(p-1)(p-2)(p-3)(p-4)(na)^4(a+bx^n)^{p-4}}\\&+ - - - - - - - - - - - - - - -\\&- - - - - - - - - - - - - - - -\\&+\frac{(n(p-1)-(m+1))(n(p-2)-(m+1)) - - - (n.2-(m+1))}{(p-1)(p-2) - - - - - - 3.2.(na)^{p-1}(a+b^n)}\Bigg)x^{m+1}\\&+\frac{(n(p-1)-(m+1))(n(p-2)-(m+1)) - - (n.1-(m+1))}{(p-1)(p-2) - - - - - - - 3.2.1.(na)^{p-1}}\int\frac{x^m\partial x}{a+bx^n}.\end{aligned}$$

$$\text{III.}\int\frac{\partial x}{(a+bx^n)^p}=\frac{x}{(p-1)na(a+bx^n)^{p-1}}+\frac{(np-n-1)x}{(p-1)(p-2)(na)^2(a+bx^n)^{p-2}}$$

$+$

$$+\frac{(np-n-1)(np-2n-1)x}{(p-1)(p-2)(p-3)(na)^3(a+bx^n)^{p-3}}$$

$$+\frac{(np-n-1)(np-2n-1)(np-3n-1)x}{(p-1)(p-2)(p-3)(p-4)(na)^4(a+bx^n)^{p-4}}$$

$$+ - - - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - - - - -$$

$$+\frac{(np-n-1)(np-2n-1) - - - (n.2-1)x}{(p-1)(p-2) - - - 3.2.(na)^{p-1}(a+bx^n)}$$

$$+\frac{(np-n-1)(np-2n-1) - - - (n.1-1)}{(p-1)(p-2) - - - - - - 3.2.1.(na)^{p-1}}\int\frac{\partial x}{a+bx^n},$$

$$\text{IV. }\int\frac{\partial x}{x^m(a+bx^n)^p}=\Big(\frac{1}{(p-1)na(a+bx^n)^{p-1}}$$

$$+\frac{n(p-1)+m-1}{(p-1)(p-2)(na)^2(a+bx^n)^{p-2}}$$

$$+\frac{(n(p-1)+m-1)(n(p-2)+m-1)}{(p-1)(p-2)(p-3)(na)^3(a+bx^n)^{p-3}}$$

$$+\frac{(n(p-1)+m-1)(n(p-2)+m-1)(n(p-3)+m-1)}{(p-1)(p-2)(p-3)(p-4)(na)^4(a+bx^n)^{p-4}}$$

$$+ - - - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - - - - -$$

$$+\frac{(n(p-1)+m-1)(n(p-2)+m-1) - - - (n.2+m-1)}{(p-1)(p-2)(p-3) - - - 3.2.(na)^{p-1}(a+bx^n)}\Big)\frac{1}{x^{m-1}}$$

$$+\frac{(n(p-1)+m-1)(n(p-2)+m-1) - - - (n.1+m-1)}{(p-1)(p-2) - - - - - - 3.2.1.(na)^{p-1}}\int\frac{\partial x}{x^m(a+bx^n)}.$$

285. Problema.

Integrare exponentes differentiales fractos $\frac{\partial x}{a+bx^n}$, $\frac{x^m\partial x}{a+bx^n}$, $\frac{\partial x}{x^m(a+bx^n)}$, $\frac{\partial x}{(a+bx^n)^q}$, $\frac{x^m\partial x}{(a+bx^n)^q}$, $\frac{\partial x}{x^m(a+bx^n)^q}$, *pro quibuscunque numeris integris et positivis* m, n, q.

Solutio.

1. Pro $n=2$ habentur integralia in (264. 266. 267. 268. 270. 271. 272. §.). Quodsi autem sit $n>2$, dispiciatur ante omnia, annon sit $\frac{m+1}{n}$ in $\frac{x^m\partial x}{a+bx^n}$, $\frac{x^m\partial x}{(a+bx^n)^q}$ numerus integer positivus, quo casu quaerantur

inte-

integralia per (282. §.), posito $p = -1$), pro primo, et $p = -q$ pro secundo exponente. Porro videatur, annon sit $\frac{m-1}{n}$ in $\frac{dx}{x^m(a+bx^n)}$, $\frac{dx}{x^m(a+bx^n)^q}$ numerus integer positivus: hoc enim casu integrabuntur isti exponentes commodissime per (283. §.).

2. Absentibus his conditionibus (1) quaerantur integralia hac ratione. Si est $b = \pm h$; invenies integralia

$$\int \frac{dx}{a+bx^n} = \frac{1}{h}\int \frac{dx}{\frac{a}{h} \pm x^n}; \int \frac{x^{\pm m}dx}{a+bx^n} = \frac{1}{h}\int \frac{x^{\pm m}dx}{\frac{a}{h} \pm x^n}$$

per (276 - - - 281. §.): inventis autem istis, determinabis integralia $\int \frac{dx}{(a+bx^n)^q}$, $\int \frac{x^{\pm m}dx}{(a+bx^n)^q}$ per (284. §.).

CAPUT V.

DE INTEGRATIONE EXPONENTIUM IRRATIONALIUM, FORMAE GENERALIS $x^m \partial x (a + b x^n \pm c x^{2n})^p$.

286. Problema.

Integrare exponentes differentiales formula generali $\frac{A \partial x}{\sqrt{(a + b x \pm c x^2)}}$ *comprehensos.*

Solutio.

Pro $z = x\sqrt{c} + \sqrt{(a + bx + cx^2)}$ obtinebitur sequens transformatio dati exponentis, sumto c cum signo $+$, unde per (252. §.) elicietur ejus integrale (I).

$$\frac{A \partial x}{\sqrt{(a + bx + cx^2)}} = 2A \cdot \frac{\partial z}{b + 2z\sqrt{c}}.$$

Si vero ponas $z = \frac{2cx - b}{\sqrt{(4ac + b^2)}}$, obtinebis sequentem transformationem dati exponentis differentialis pro c cum signo $-$, unde per (240. §.) reperietur ejus integrale (II).

$$\frac{A \partial x}{\sqrt{(a + bx - cx^2)}} = \frac{A}{\sqrt{c}} \cdot \frac{\partial z}{\sqrt{(1 - z^2)}}.$$

$$\text{I.} \int \frac{A \partial x}{\sqrt{(a + bx + cx^2)}} = \frac{A}{\sqrt{c}}\, l \left(2cx + b + 2\sqrt{c}\sqrt{(a + bx + cx^2)}\right) + C.$$

$$\text{II.} \int \frac{A \partial x}{\sqrt{(a + bx - cx^2)}} = \frac{A}{\sqrt{c}} \text{ Arc Sin } \frac{2cx - b}{\sqrt{(4ac + b^2)}} + C.$$

$$= \frac{A}{\sqrt{c}} \text{ Arc Tang } \frac{2cx - b}{2\sqrt{c}\sqrt{(a + bx - cx^2)}} + C.$$

287. Corollarium 1.

Pro $b = 0$, et $a = 0$ seorsim prodibunt hinc (286. §.) sequentes formulae integrales,

$$\int\frac{A\,\partial x}{\sqrt{(a+cx^2)}}=\frac{A}{\sqrt{c}}\,l\left(x\sqrt{c}+\sqrt{(a+cx^2)}\right)+C.$$

$$\int\frac{A\,\partial x}{\sqrt{(a-cx^2)}}=\frac{A}{\sqrt{c}}\text{ Arc Sin }\frac{x\sqrt{c}}{\sqrt{a}}+C.$$

$$\int\frac{A\,\partial x}{x^{\frac{1}{2}}\sqrt{(b+cx)}}=\frac{A}{\sqrt{c}}\,l\left(2cx+b+2\sqrt{c}\sqrt{(bx+cx^2)}\right)+C.$$

$$\int\frac{A\,\partial x}{x^{\frac{1}{2}}\sqrt{(b-cx)}}=\frac{A}{\sqrt{c}}\text{ Arc Sin }\frac{2cx-b}{b}+C.$$

288. Corollarium 2.

In (262. §. II.) ponatur $q=\frac{1}{2}$, obtinebitur inde $\int\partial x\sqrt{X}$ $=\frac{(2\gamma x+\beta)\sqrt{X}}{4\gamma}+\frac{4\alpha\gamma-\beta^2}{8\gamma}\int\frac{\partial x}{\sqrt{X}}$, pro $X=\alpha+\beta x+\gamma x^2$: quodsi ergo fiat $\alpha=a$, $\beta=b$, $\gamma=\pm c$, reperientur per (286. §.) sequentia integralia.

$$\int\partial x\sqrt{(a+bx+cx^2)}=\frac{(2cx+b)\sqrt{(a+bx+cx^2)}}{4c}$$
$$+\frac{4ac-b^2}{8c\sqrt{c}}\,l(2cx+b+2\sqrt{c}\sqrt{(a+bx+cx^2)})+C.$$

$$\int\partial x\sqrt{(a+bx-cx^2)}=\frac{(2cx-b)\sqrt{(a+bx-cx^2)}}{4c}$$
$$+\frac{4ac+b^2}{8c\sqrt{c}}\text{ Arc Sin }\frac{2cx-b}{\sqrt{(4ac+b^2)}}+C$$
$$=\frac{(2cx-b)\sqrt{(a+bx-cx^2)}}{4c}$$
$$+\frac{4ac+b^2}{8c\sqrt{c}}\text{ Arc Tang }\frac{2cx-b}{2\sqrt{c}\sqrt{(a+bx-cx^2)}}+C.$$

289. Corollarium 3.

Hinc autem pro $b=0$, et $a=0$ seorsim obtinebuntur sequentes formulae integrales.

$$\int\partial x\sqrt{(a+cx^2)}=\tfrac{1}{2}x\sqrt{(a+cx^2)}+\frac{a}{2\sqrt{c}}\,l(x\sqrt{c}+\sqrt{(a+cx^2)})+C.$$

$$\int\partial x\sqrt{(a-cx^2)}=\tfrac{1}{2}x\sqrt{(a-cx^2)}+\frac{a}{2\sqrt{c}}\text{ Arc Sin }\frac{x\sqrt{c}}{\sqrt{a}}+C.$$

$$\int x^{\frac{1}{2}}\partial x\sqrt{(b+cx)}=\frac{(2cx+b)\sqrt{(bx+cx^2)}}{4c}$$
$$-\frac{b^2}{8c\sqrt{c}}\,l(2cx+b+2\sqrt{c}\sqrt{(bx+cx^2)})+C.$$
$$\int x^{\frac{1}{2}}\partial x\sqrt{(b-cx)}=\frac{(2cx-b)\sqrt{(bx-cx^2)}}{4c}$$
$$+\frac{b^2}{8c\sqrt{c}}\text{ Arc Sin }\frac{2cx-b}{b}+C.$$

290. Corollarium 4.

Exponens $\frac{A\partial x}{x\sqrt{(cx^2+bx\pm a)}}$ pro $z=\frac{\pm 2a+bx}{\pm 2ax}$ abibit in $\frac{-A\partial z}{\sqrt{\left(\frac{\pm 4ac-b^2}{\pm 4a}\pm az^2\right)}}$: quamobrem per (287. §.) reperientur sequentes formulae integrales.

$$\int\frac{A\partial x}{x\sqrt{(cx^2+bx+a)}}=C-\frac{A}{\sqrt{a}}\,l\left(\frac{2a+bx+2\sqrt{a}\sqrt{(cx^2+bx+a)}}{x}\right).$$
$$\int\frac{A\partial x}{x\sqrt{(cx^2+bx-a)}}=\frac{A}{\sqrt{a}}\text{ Arc Cosin }\frac{2a-bx}{x\sqrt{(4ac+b^2)}}+C.$$
$$=\frac{A}{\sqrt{a}}\text{ Arc Tang }\frac{2\sqrt{a}\sqrt{(cx^2+bx-a)}}{2a-bx}+C.$$

291. Corollarium 5.

Et hinc, pro $b=0$, $c=0$ seorsim, fluunt sequentes formulae integrales.

$$\int\frac{A\partial x}{x\sqrt{(cx^2+a)}}=C-\frac{A}{\sqrt{a}}\,l\left(\frac{\sqrt{a}+\sqrt{(cx^2+a)}}{x}\right).$$
$$\int\frac{A\partial x}{x\sqrt{(cx^2-a)}}=C+\frac{A}{\sqrt{a}}\text{ Arc Cosin }\frac{\sqrt{a}}{x\sqrt{c}}.$$
$$\int\frac{A\partial x}{x\sqrt{(bx+a)}}=C-\frac{A}{\sqrt{a}}\,l\left(\frac{2a+bx+2\sqrt{a}\sqrt{(bx+a)}}{x}\right).$$
$$\int\frac{A\partial x}{x\sqrt{(bx-a)}}=C+\frac{A}{\sqrt{a}}\text{ Arc Cosin }\frac{2a-bx}{bx}.$$

292. Corollarium 6.

Posito $z = bx \pm a$ fiet $\frac{x^{\frac{1}{2}} \partial x}{bx \pm a} = \frac{\partial z \sqrt{(z \mp a)}}{z b^{\frac{3}{2}}}$, adeoque $\frac{x^{\frac{1}{2}} \partial x}{bx \pm a}$
$= \frac{1}{b^{\frac{3}{2}}} \cdot \frac{\partial z}{\sqrt{(z \mp a)}} \mp \frac{a}{b^{\frac{3}{2}}} \cdot \frac{\partial z}{z \sqrt{(z \mp a)}}$.

Eapropter, si integrale algebraicum primae partis capiatur per (241. 248. §.), et pars altera integretur per (291. §.). prodibunt sequentia integralia.

$$\int \frac{x^{\frac{1}{2}} \partial x}{bx - a} = \frac{2x^{\frac{1}{2}}}{b} - \frac{\sqrt{a}}{b\sqrt{b}}\, l\left(\frac{bx + a + 2\sqrt{abx}}{bx - a}\right) + C.$$

$$\int \frac{x^{\frac{1}{2}} \partial x}{bx + a} = \frac{2x^{\frac{1}{2}}}{b} - \frac{\sqrt{a}}{b\sqrt{b}} \text{ Arc Cosin} \frac{a - bx}{a + bx} + C.$$

293. Problema.

Integrare exponentes differentiales $\partial x \sqrt{X^k}$, $\frac{\partial x}{\sqrt{X^k}}$, $\frac{\partial x \sqrt{X^k}}{x}$, $\frac{\partial x}{x\sqrt{X^k}}$ *pro quovis numero integro impari et positivo* k, *et quolibet trinomio* $\alpha + \beta x + \gamma x^2 = X$.

Solutio.

Ponatur in (262. §. II.) $q = \pm \frac{n}{2}$; invenietur sequens formula (I) et (II). Porro fiat in (261. §. III.) $m = 0$ et $p = \pm \frac{n}{2}$; obtinebitur sequens formula (III) et (IV). Cum pro $n = 3$ pars secunda primae formulae fiat aequalis nihilo; erit integrale exponentis differentialis $\frac{\partial x}{\sqrt{X^k}}$ pro quovis numero impari positivo $k > 1$ perfecte algebraicum, licet id pro $k = 1$ fit logarithmicum, aut trigonometricum (286. §.): invenietur vero illud ope formulae (I), si in hac fiat successive $n = 3$, $n = 5$, $n = 7$ - - - $n = k$. Secunda autem formula, si in ea ponatur successive $n = 1$, $n = 3$, $n = 5$, - - - $n = k - 2$, dabit integrale $\int \partial x \sqrt{X^k}$ pro quovis numero impari positivo k dependenter a noto integrali $\int \partial \sqrt{X}$ (288. §.). Pari ratione, si fiat $n = -1$, $n = 1$, $n = 3$, $n = 5$. - - - $n = k - 2$, invenietur integrale $\int \frac{\partial x \sqrt{X^k}}{x}$ ope tertiae formulae dependenter ab notis integralibus $\int \frac{\partial x}{\sqrt{X}}$, $\int \frac{\partial x}{x\sqrt{X}}$ (286. 290. §.). Formula quarta demum dabit integrale

 expo-

exponentis $\frac{\varepsilon x}{x\sqrt{X^k}}$ dependenter a notis integralibus $\frac{\varepsilon x}{x\sqrt{X}}$ (290. §), $\frac{\varepsilon x}{\sqrt{X^3}}$, $\frac{\varepsilon x}{\sqrt{X^5}}$ etc., si successive fiat $n=3$, $n=5$, $n=7$, $---n=k$.

$$\text{I. } \int\frac{\varepsilon x}{\sqrt{X^n}} = \frac{2(2\gamma x+\beta)}{(n-2)(4\alpha\gamma-\beta^2)\sqrt{X^{n-2}}} + \frac{4\gamma(n-3)}{(n-2)(4\alpha\gamma-\beta^2)}\int\frac{\varepsilon x}{\sqrt{X^{n-2}}}.$$

$$\text{II. } \int\varepsilon x\sqrt{X^{n+2}} = \frac{(2\gamma x+\beta)\sqrt{X^{n+2}}}{2g(n+3)} + \frac{(n+2)(4\alpha\gamma-\beta^2)}{4\gamma(n+3)}\int\varepsilon x\sqrt{X^n}.$$

$$\text{III. } \int\frac{\varepsilon x\sqrt{X^{n+2}}}{x} = \frac{\sqrt{X^{n+2}}}{n+2} + \alpha\int\frac{\varepsilon x\sqrt{X^n}}{x} + \tfrac{1}{2}\beta\int\varepsilon x\sqrt{X^n}.$$

$$\text{IV. } \int\frac{\varepsilon x}{x\sqrt{X^n}} = \frac{1}{(n-2)\alpha\sqrt{X^{n-2}}} - \frac{\beta}{2\alpha}\int\frac{\varepsilon x}{\sqrt{X^n}} + \frac{1}{\alpha}\int\frac{\varepsilon x}{x\sqrt{X^{n-2}}}.$$

294. Problema.

Integrare exponentes differentiales $x^r\varepsilon x\sqrt{X}$, $\frac{x^r\varepsilon x}{\sqrt{X}}$, $\frac{\varepsilon x\sqrt{X}}{x^r}$, $\frac{\varepsilon x}{x^r\sqrt{X}}$, *pro quovis trinomio* $X=\alpha+\beta x+\gamma x^2$, *et quovis numero integro positivo* r.

Solutio.

Pone $p=\frac{1}{2}$ in (261. §. I.), obtinebis sequentem formulam (I); hinc autem pro $-m$ loco m elicies formulam (II). Porro fiat in (261. §. I.) $p=-\frac{1}{2}$; prodibit inde sequens formula (III), et ex hac formula (IV) pro $-m$ loco m. Ope harum formularum facile jam reducentur quaesita integralia ad integralia $\int\varepsilon x\sqrt{X}$, $\int\frac{\varepsilon x}{\sqrt{X}}$, $\int\frac{\varepsilon x}{x\sqrt{X}}$, quae in (288. 286. 290. §.) perfecte determinavimus, et integrale $\int\frac{\varepsilon x\sqrt{X}}{x}$ per (293. §. III.) pro $n=-1$ determinabile.

$$\text{I. } \int x^{m+1}\varepsilon x\sqrt{X} = \frac{x^m\sqrt{X^3}}{\gamma(m+3)} - \frac{m\alpha}{\gamma(m+3)}\int x^{m-1}\varepsilon x\sqrt{X} - \frac{\beta(2m+3)}{2\gamma(m+3)}\int x^m\varepsilon x\sqrt{X}.$$

$$\text{II. } \int\frac{\varepsilon x\sqrt{X}}{x^m} = -\frac{\sqrt{X^3}}{(m-1)\alpha x^{m-1}} - \frac{\gamma(m-4)}{(m-1)\alpha}\int\frac{\varepsilon x\sqrt{X}}{x^{m-2}} - \frac{\beta(2m-5)}{2(m-1)\alpha}\int\frac{\varepsilon x\sqrt{X}}{x^{m-1}}.$$

$$\text{III. } \int\frac{x^{m+1}\varepsilon x}{\sqrt{X}} = \frac{x^m\sqrt{X}}{\gamma(m+1)} - \frac{m\alpha}{\gamma(m+1)}\int\frac{x^{m-1}\varepsilon x}{\sqrt{X}} - \frac{\beta(2m+1)}{2\gamma(m+1)}\int\frac{x^m\varepsilon x}{\sqrt{X}}.$$

$$\text{IV. } \int\frac{\varepsilon x}{x^m\sqrt{X}} = \frac{-\sqrt{X}}{(m-1)\alpha x^{m-1}} - \frac{\beta(2m-3)}{2(m-1)\alpha}\int\frac{\varepsilon x}{x^{m-1}\sqrt{X}} - \frac{\gamma(m-2)}{(m-1)\alpha}\int\frac{\varepsilon x}{x^{m-2}\sqrt{X}}.$$

295. Problema.

Integrare exponentes differentiales $x^r \varepsilon x \sqrt{X^k}$, $\frac{x^r \varepsilon x}{\sqrt{X^k}}$, $\frac{\varepsilon x}{x^r \sqrt{X^k}}$, $\frac{\varepsilon x \sqrt{X^k}}{x^r}$ *pro quovis trinomio* $\alpha + \beta x + \gamma x^2 = X$, *et quibuslibet numeris integris positivis* r, *imparibusque* k.

Solutio.

In (261. §. I.) fiat $p = \frac{k}{2}$, nascetur sequens formula (I), et (IV) pro $-m$ loco m. Porro pro $-n$ loco n in (I) obtinebitur formula (II), et (III) si sumatur etiam $-m$ loco m. Denique in (261. §. III.) fiat $p = -\frac{k}{2}$, et $m = n - 1$; elicietur inde sequens formula (V). Quamobrem, invento integrali $\int \varepsilon x \sqrt{X^k}$ per (293. §.), determinabis $\int x^r \varepsilon x \sqrt{X^k}$ ope primae formulae, si pro $n = k$ successive ponas $m = 0$, $m = 1$, $m = 2$, $m = 3$, --- $m = r - 1$: captis autem integralibus $\int \frac{\varepsilon x}{\sqrt{X^k}}$, $\int \frac{\varepsilon x}{x \sqrt{X^k}}$ per (293. §.), invenies integrale $\int \frac{\varepsilon x}{x^r \sqrt{X^k}}$ ope tertiae formulae, si in illa pro $n = k$ successive ponas $m = 1$, $m = 2$, $m = 3$, --- $m = r - 1$. Porro determinatis integralibus $\int \varepsilon x \sqrt{X^k}$. $\int \frac{\varepsilon x \sqrt{X^k}}{x}$ per (293. §.) invenies eadem ratione integrale $\int \frac{\varepsilon x \sqrt{X^k}}{x^r}$ ope quartae formulae. Quod autem ad integrationem exponentis differentialis $\frac{x^r \varepsilon x}{\sqrt{X^k}}$ adtinet; dispiciatur, an sit $r < k - 1$, quo casu invenietur ejus integrale ope solius secundae formulae, si in hac pro $n = k$ successive ponatur $m = 0$, $m = 1$, $m = 2$, $m = 3$, - - - - $m = r - 1$: unde simul constat, integrale hoc futurum perfecte algebraicum, dependens ab integrali algebraico $\int \frac{\varepsilon x}{\sqrt{X^k}}$ per (293. §.) determinabili. Quodsi autem fuerit $r = k - 1$, vel $r > k - 1$, debebit formula (II) cum (V) conjungi, ita ut ope formulae (V) ea integralia definiantur, quae formula (II) existente $m = n - 2$ dare nequiverit, integralia autem, quibus in adplicatione formulae (V) opus fuerit, ope formulae (II) determinentur.

$$\text{I.} \int x^{m+1} \varepsilon x \sqrt{X^n} = \frac{x^m \sqrt{X^{n+2}}}{\gamma(n+2+m)} - \frac{m\alpha}{\gamma(n+2+m)} \int x^{m-1} \varepsilon x \sqrt{X^n}$$
$$- \frac{\beta(n+2-2m)}{2\gamma(n+2+m)} \int x^m \varepsilon x \sqrt{X^n}.$$

II.

$$\text{II.} \int \frac{x^{m+1}\partial x}{\sqrt{X^n}} = \frac{x^m}{\gamma(m+2-n)\sqrt{X^{n-2}}} - \frac{m\alpha}{\gamma(m+2-n)}\int\frac{x^{m-1}\partial x}{\sqrt{X^n}} - \frac{\beta(2m+2-n)}{2\gamma(m+2-n)}\int\frac{x^m\partial x}{\sqrt{X^n}}.$$

$$\text{III.} \int \frac{\partial x}{x^{m+1}\sqrt{X^n}} = \frac{-1}{m\alpha x^m\sqrt{X^{n-2}}} - \frac{\beta(2m+n-2)}{2m\alpha}\int\frac{\partial x}{x^m\sqrt{X^n}} - \frac{\gamma(m+n-2)}{m\alpha}\int\frac{\partial x}{x^{m-1}\sqrt{X^n}}.$$

$$\text{IV.} \int \frac{\partial x\sqrt{X^n}}{x^{m+1}} = -\frac{\sqrt{X^{n+2}}}{m\alpha x^m} + \frac{\gamma(n+2-m)}{m\alpha}\int\frac{\partial x\sqrt{X^n}}{x^{m-1}} + \frac{\beta(n+2-2m)}{2m\alpha}\int\frac{\partial x\sqrt{X^n}}{x^m}.$$

$$\text{V.} \int \frac{x^{n-1}\partial x}{\sqrt{X^n}} = \frac{2x^{n-1}}{\beta(n-2)\sqrt{X^{n-2}}} - \frac{2\alpha}{\beta}\int\frac{x^{n-2}\partial x}{\sqrt{X^n}} - \frac{2}{\beta(n-2)}\int\frac{x^{n-2}\partial x}{\sqrt{X^{n-2}}}.$$

296. Corollarium.

Pro $z = x^k$ transformabitur exponens differentialis $x^r\partial x(\alpha+\beta x^k+\gamma x^{2k})^{\frac{q}{2}}$ in $\frac{1}{k}\cdot z^{\frac{r+1}{k}-1}\cdot\partial z\sqrt{(\alpha+\beta z+\gamma z^2)^q}$: omnes igitur exponentes differentiales illa formula comprehensi perfecte erunt integrabiles, modo fit $\frac{r+1}{k}$ numerus integer, positivus vel negativus.

297. Problema.

Dato integrali $\int x^{m-1}\partial x(\alpha+\beta x)^p$ *invenire* $\int x^m\partial x(\alpha+\beta x)^p$, *et dato hoc invenire* $\int x^m\partial x(\alpha+\beta x)^{p+1}$.

Solutio.

Primum integrale, ut sequitur in (I), obtinebitur ex (261. §. II.), posito $\gamma=0$. Quodsi porro idem ejus valor substituatur in (261. §. III.), tum scribatur $m+1$ loco m; prodibit inde integrale (II.).

$$\text{I.} \int x^m\partial x(\alpha+\beta x)^p = \frac{x^m(\alpha+\beta x)^{p+1}}{\beta(m+p+1)} - \frac{m\alpha}{\beta(m+p+1)}\int x^{m-1}\partial x(\alpha+\beta x)^p.$$

$$\text{II.} \int x^m\partial x(\alpha+\beta x)^{p+1} = \frac{x^{m+1}(\alpha+\beta x)^{p+1}}{m+p+2} + \frac{\alpha(p+1)}{m+p+2}\int x^m\partial x(\alpha+\beta x)^p.$$

298. Corollarium 1.

Prima formula pro $-m$ loco m dabit sequentem (I); secunda vero formula pro $-p$ loco p dabit sequentem (II).

$$\text{I. } \int\frac{\partial x(\alpha+\beta x)^p}{x^{m+1}} = -\frac{(\alpha+\beta x)^{p+1}}{m\alpha x^m} + \frac{\beta(p+1-m)}{m\alpha}\int\frac{\partial x(\alpha+\beta x)^p}{x^m}.$$

$$\text{II. } \int\frac{x^m\partial x}{(\alpha+\beta x)^p} = \frac{x^{m+1}}{\alpha(p-1)(\alpha+\beta x)^{p-1}} - \frac{(m-p+2)}{\alpha(p-1)}\int\frac{x^m\partial x}{(\alpha+\beta x)^{p-1}}.$$

299. Corollarium 2.

Dato integrali $\frac{x^{\frac{1}{2}}\partial x}{\alpha+\beta x}$ (292. §.) determinabuntur sequentia integralia pro quovis numero integro positivo q, et quovis impari n, si in (297. §. I.) pro $p=-1$ fiat successive $m=\frac{3}{2}$, $m=\frac{5}{2}$, $m=\frac{7}{2}$, - - - - $m=\frac{n}{2}$, et in (298. §. II.) pro $m=\frac{n}{2}$ fiat successive $p=2$, $p=3$, $p=4$, - - - $p=q$.

$$\int\frac{x^{\frac{n}{2}}\partial x}{\alpha+\beta x} = \frac{2x^{\frac{n}{2}}}{n\beta} - \frac{2\alpha x^{\frac{n-2}{2}}}{(n-2)\beta^2} + \frac{2\alpha^2 x^{\frac{n-4}{2}}}{(n-4)\beta^3}$$

$$-\frac{2\alpha^3 x^{\frac{n-6}{2}}}{(n-6)\beta^4} + \frac{2\alpha^4 x^{\frac{n-8}{2}}}{(n-8)\beta^5} - \text{- - - -} + \text{- - - - -}$$

$$\pm\frac{2\alpha^{\frac{n-3}{2}}\cdot x^{\frac{1}{2}}}{3\beta^{\frac{n-1}{2}}} \mp \frac{\alpha^{\frac{n-1}{2}}}{\beta^{\frac{n-1}{2}}}\int\frac{x^{\frac{1}{2}}\partial x}{\alpha+\beta x}.$$

$$\int\frac{x^{\frac{n}{2}}\partial x}{(\alpha+\beta x)^q} = \Big(\frac{1}{(q-1)\alpha(\alpha+\beta x)^{q-1}} - \frac{n-2q+4}{2(q-1)(q-2)\alpha^2(\alpha+\beta x)^{q-2}}$$

$$+\frac{(n-2q+4)(n-2q+6)}{2^2(q-1)(q-2)(q-3)\alpha^3(\alpha+\beta x)^{q-3}} - \text{- - - - - -}$$

$$\pm\frac{(n-2q+4)(n-2q+6) \text{ - - - } (n-4)(n-2)}{2^{q-2}\cdot(q-1)(q-2) \text{ - - - } 3\cdot 2\cdot 1\cdot\alpha^{q-1}(\alpha+\beta x)}\Big)x^{\frac{n+2}{2}}$$

$$\mp\frac{(n-2q+4)(n-2q+6) \text{ - - - } (n-2)n}{2^{q-1}\cdot(q-1)(q-2) \text{ - - - - } 3\cdot 2\cdot 1\cdot\alpha^{q-1}}\int\frac{x^{\frac{1}{2}}\partial x}{\alpha+\beta x}.$$

300. Corollarium 3.

Per idem integrale (292. §.) determinabuntur quoque sequentia integralia pro numeris integris positivis p, q, si in (298. §.) sumatur prima formu-

formula, et in ea pro $p=-1$ fiat successive $m=-\frac{1}{2}$, $m=\frac{1}{2}$, $m=\frac{3}{2}$, --- $m=\frac{n}{2}-1$, in secunda autem formula ejusdem §phi pro $m=-\frac{n}{2}$ successive ponatur $p=2$, $p=3$, $p=4$, --- $p=q$.

$$\int\frac{\partial x}{x^{\frac{n}{2}}(\alpha+\beta x)}=\frac{-2}{(n-2)\alpha x^{\frac{n-2}{2}}}+\frac{2\beta}{(n-4)\alpha^2 x^{\frac{n-4}{2}}}-\frac{2\beta^2}{(n-6)\alpha^3 x^{\frac{n-6}{2}}}+\text{-------}$$

$$\pm\frac{2\beta^{\frac{n-3}{2}}}{\alpha^{\frac{n-1}{2}}.x^{\frac{1}{2}}}\pm\frac{2\beta^{\frac{n-1}{2}}.x^{\frac{1}{2}}}{\alpha^{\frac{n+1}{2}}}\mp\frac{\beta^{\frac{n+1}{2}}}{\alpha^{\frac{n+1}{2}}}\int\frac{x^{-\frac{1}{2}}\partial x}{\alpha+\beta x}.$$

et

$$\int\frac{\partial x}{x^{\frac{n}{2}}(\alpha+\beta x)^q}=\Bigg(\frac{1}{(q-1)\alpha(\alpha+\beta x)^{q-1}}+\frac{n+2q-4}{2(q-1)(q-2)\alpha^2(\alpha+\beta x)^{q-2}}$$

$$+\frac{(n+2q-4)(n+2q-6)}{2^2(q-1)(q-2)(q-3)\alpha^3(\alpha+\beta x)^{q-3}}+\text{------}$$

$$+\frac{(n+2q-4)(n+2q-6)\text{ --- }(n+4)(n+2)}{2^{q-2}.(q-1)(q-2)\text{ --- }3.2.\alpha^{q-1}(\alpha+\beta x)}\Bigg)\frac{1}{x^{\frac{n-2}{2}}}$$

$$+\frac{(n+2q-4)(n+2q-6)\text{ --- }(n+4)(n+2)n}{2^{q-1}.(q-1)(q-2)\text{ --- }3.2.\alpha^{q-1}}\int\frac{\partial x}{x^{\frac{n}{2}}(\alpha+\beta x)}.$$

301. Corollarium 4.

Duo sequentia integralia determinata per notum integrale $x^{\frac{1}{2}}\partial x\sqrt{(\alpha+\beta x)}$ (289. §.) derivabuntur ex (297. §.), si in prima formula pro $p=\frac{1}{2}$ fiat $m=\frac{3}{2}$, $m=\frac{5}{2}$, --- $m=\frac{n}{2}-1$, et in secunda formula $p=\frac{1}{2}$, $p=\frac{3}{2}$, --- $p=\frac{n}{2}-1$ pro $m=\frac{n}{2}$.

$$\int x^{\frac{n}{2}}\partial x\sqrt{(\alpha+\beta x)}=\Bigg(\frac{x^{\frac{n}{2}}}{(n+3)\beta}-\frac{n\alpha x^{\frac{n-2}{2}}}{(n+3)(n+1)\beta^2}$$

$$+\frac{n(n-2)\alpha^2.x^{\frac{n-4}{2}}}{(n+3)(n+1)(n-1)\beta^3}-\text{------}+\text{------}$$

$$\pm\frac{n(n-2)(n-4)\text{ --- }7.5\alpha^{\frac{n-3}{2}}.x^{\frac{3}{2}}}{(n+3)(n+1)(n-1)\text{ --- }8.6.\beta^{\frac{n-1}{2}}}\Bigg)2\sqrt{(\alpha+\beta x)^3}$$

$$\mp\frac{n(n-2)(n-4)\text{ --- }5.3\alpha^{\frac{n-1}{2}}}{(n+3)(n+1)(n-1)\text{ --- }8.6.\beta^{\frac{n-1}{2}}}\int x^{\frac{1}{2}}\partial x\sqrt{(\alpha+\beta x)}.$$

$$\int x^{\frac{n}{2}}\partial x\sqrt{(\alpha+\beta x)^q}=\Big(\frac{\sqrt{(\alpha+\beta x)^q}}{n+q+2}+\frac{q\alpha\sqrt{(\alpha+\beta x)^{q-2}}}{(n+q+2)(n+q)}$$
$$+\frac{q(q-2)\alpha^2\sqrt{(\alpha+\beta x)^{q-4}}}{(n+q+2)(n+q)(n+q-2)}+\text{-------}$$
$$+\frac{q(q-2)(q-4)\text{---}7.5.\alpha^{\frac{q-3}{2}}.\sqrt{(\alpha+\beta x)^3}}{(n+q+2)(n+q)(n+q-2)\text{---}(n+7)(n+5)}\Big)2x^{\frac{n+2}{2}}$$
$$+\frac{q(q-2)(q-4)\text{----}7.5.3.\alpha^{\frac{q-1}{2}}}{(n+q+2)(n+q)(n+q-2)\text{--}(n+7)(n+5)}\int x^{\frac{n}{2}}\partial x\sqrt{(\alpha+\beta x)}.$$

302. Corollarium 5.

Ex (297. §. I.) pro $p=-\frac{1}{2}$ et $m=\frac{1}{2}$, $m=\frac{3}{2}$, --- $m=\frac{n}{2}$ derivabitur sequens primum integrale per notum integrale $\int\frac{\partial x}{x^{\frac{1}{2}}\sqrt{(\alpha+\beta x)}}$ (287. §.) perfecte determinatum, per id ipsum vero determinabitur sequens secundum integrale, si in (298. §. II.) pro $m=\frac{n}{2}$ fiat $p=\frac{3}{2}$, $p=\frac{5}{2}$, --- $p=\frac{q}{2}$.

$$\int\frac{x^{\frac{n}{2}}\partial x}{\sqrt{(\alpha+\beta x)}}=\Big(\frac{x^{\frac{n}{2}}}{(n+1)\beta}-\frac{n\alpha x^{\frac{n-2}{2}}}{(n+1)(n-1)\beta^2}+\frac{n(n-2)\alpha^2x^{\frac{n-4}{2}}}{(n+1)(n-1)(n-3)\beta^3}$$
$$-\text{-----------}+\text{---------}$$
$$\pm\frac{n(n-2)(n-4)\text{---}5.3\,\alpha^{\frac{n-1}{2}}.x^{\frac{1}{2}}}{(n+1)(n-1)(n-3)\text{---}4.2.\beta^{\frac{n+1}{2}}}\Big)2\sqrt{(\alpha+\beta x)}$$
$$\mp\frac{n(n-2)(n-4)\text{---}5.3.1.\alpha^{\frac{n+1}{2}}}{(n+1)(n-1)(n-3)\text{---}4.2.\beta^{\frac{n+1}{2}}}\int\frac{\partial x}{x^{\frac{1}{2}}\sqrt{(\alpha+\beta x)}}.$$

$$\int\frac{x^{\frac{n}{2}}\partial x}{\sqrt{(\alpha+\beta x)^q}}=\Big(\frac{1}{(q-2)\alpha\sqrt{(\alpha+\beta x)^{q-2}}}-\frac{n-q+4}{(q-2)(q-4)\alpha^2\sqrt{(\alpha+\beta x)^{q-4}}}$$
$$+\frac{(n-q+4)(n-q+6)}{(q-2)(q-4)(q-6)\alpha^3\sqrt{(\alpha+\beta x)^{q-6}}}\text{----}+\text{----}$$
$$\pm\frac{(n-q+4)(n-q+6)\text{----}(n-1)}{(q-2)(q-4)\text{---}3.1.\alpha^{\frac{q-1}{2}}.\sqrt{(\alpha+\beta x)}}\Big)2x^{\frac{n+2}{2}}$$
$$\mp\frac{(n-q+4)(n-q+6)\text{---}(n-1)(n+1)}{(q-2)(q-4)\text{-------}3.1.\alpha^{\frac{q-1}{2}}}\int\frac{x^{\frac{n}{2}}\partial x}{\sqrt{(\alpha+\beta x)}}.$$

Erit autem hoc alterum integrale perfecte algebraicum, independens a primo integrali, quoties fuerit $q = n+4$, vel $q > n+4$.

303. Corollarium 6.

Per idem integrale $\int \frac{\partial x}{x^{\frac{1}{2}} \sqrt{(\alpha+\beta x)}}$ notum ex (287. §.) determinabuntur quoque duo sequentia integralia, si in (297. §. II.) pro $m = -\frac{n}{2}$ fiat $p = -\frac{1}{2}$, $p = \frac{1}{2}$, $p = \frac{3}{2}$, ---- $p = \frac{q}{2} - 1$; et in (298. §. I.) pro $p = \frac{q}{2}$ ponatur $m = \frac{1}{2}$, $m = \frac{3}{2}$, --- $m = \frac{n}{2} - 1$.

$$\int \frac{\partial x \sqrt{(\alpha+\beta x)^q}}{x^{\frac{1}{2}}} = \Big(\frac{\sqrt{(\alpha+\beta x)^q}}{q+1} + \frac{q\alpha \sqrt{(\alpha+\beta x)^{q-2}}}{(q+1)(q-1)}$$

$$+ \frac{q(q-2)\alpha^2 \sqrt{(\alpha+\beta x)^{q-4}}}{(q+1)(q-1)(q-3)} + \text{-------}$$

$$+ \frac{q(q-2)(q-4) \text{---} 5.3.\alpha^{\frac{q-1}{2}}.\sqrt{(\alpha+\beta x)}}{(q+1)(q-1)(q-3) \text{----} 4.2} \Big) 2x^{\frac{1}{2}}$$

$$+ \frac{q(q-2)(q-4) \text{---} 5.3.1.\alpha^{\frac{q+1}{2}}}{(q+1)(q-1)(q-3) \text{---} 4.2} \int \frac{\partial x}{x^{\frac{1}{2}} \sqrt{(\alpha+\beta x)}}.$$

$$\int \frac{\partial x \sqrt{(\alpha+\beta x)^q}}{x^{\frac{n}{2}}} = -\Big(\frac{1}{(n-2)\alpha x^{\frac{n-2}{2}}} + \frac{(q-n+4)\beta}{(n-2)(n-4)\alpha^2 x^{\frac{n-4}{2}}}$$

$$+ \frac{(q-n+4)(q-n+6)\beta^2}{(n-2)(n-4)(n-6)\alpha^3 . x^{\frac{n-6}{2}}} + \text{------}$$

$$+ \frac{(q-n+4)(q-n+6) \text{---} (q-1)\beta^{\frac{n-3}{2}}}{(n-2)(n-4) \text{---} 3.1.\alpha^{\frac{n-1}{2}}. x^{\frac{1}{2}}} \Big) 2\sqrt{(\alpha+\beta x)^{q+2}}$$

$$+ \frac{(q-n+4)(q-n+6) \text{---} (q-1)(q+1)\beta^{\frac{n-1}{2}}}{(n-2)(n-4) \text{------} 3.1.\alpha^{\frac{n-1}{2}}} \int \frac{\partial x \sqrt{(\alpha+\beta x)^q}}{x^{\frac{1}{2}}}$$

Ceterum per se patet, hoc alterum integrale futurum perfecte algebraicum, independens a primo integrali, si fuerit $n = q+4$ vel $n > q+4$.

304. Corollarium 7.

Ex iisdem formulis (297. §. II. et 298. §. I.) determinabuntur per integrale $\int \frac{\partial x}{x\sqrt{(\alpha+\beta x)}}$ ex (292. §.) notum, sequentia integralia, si in priori formula

mula pro $m = -1$ fiat $p = -\frac{1}{2}$, $p = \frac{1}{2}$, $p = \frac{3}{2}$, --- $p = \frac{q}{2} - 1$, et in posteriori formula pro $p = \frac{q}{2}$ fiat $m = 1$, $m = 2$, $m = 3$, --- $m = n - 1$.

$$\int \frac{dx\sqrt{(\alpha+\beta x)^q}}{x} = \frac{2\sqrt{(\alpha+\beta x)^q}}{q} + \frac{2\alpha\sqrt{(\alpha+\beta x)^{q-2}}}{q-2}$$
$$+ \frac{2\alpha^2\sqrt{(\alpha+\beta x)^{q-4}}}{q-4} + \cdots\cdots$$
$$+ \frac{2\alpha^{\frac{q-1}{2}}\sqrt{(\alpha+\beta x)}}{1} + \alpha^{\frac{q+1}{2}} \int \frac{dx}{x\sqrt{(\alpha+\beta x)}}.$$

$$\int \frac{dx\sqrt{(\alpha+\beta x)^q}}{x^n} = -\Big(\frac{1}{(n-1)\alpha x^{n-1}} + \frac{(q-2n+4)\beta}{2(n-1)(n-2)\alpha^2 . x^{n-2}}$$
$$+ \frac{(q-2n+4)(q-2n+6)\beta^2}{2^2(n-1)(n-2)(n-3)\alpha^3 . x^{n-3}} + \cdots\cdots$$
$$+ \frac{(q-2n+4)(q-2n+6)\cdots(q-2)\beta^{n-2}}{2^{n-2}.(n-1)(n-2)\cdots 3.2.1.\alpha^{n-1}.x}\Big)\sqrt{(\alpha+\beta x)^{q+2}}$$
$$+ \frac{(q-2n+4)(q-2n+6)\cdots(q-2)q\beta^{n-1}}{2^{n-1}.(n-1)(n-2)\cdots 3.2.1.\alpha^{n-1}} \int \frac{dx\sqrt{(\alpha+\beta x)^q}}{x}.$$

305. Corollarium 8.

Denique per dem. notum integrale (291. §.) determinabuntur etiam sequentia integralia, si in (298. § II.) pro $m = -1$ fiat $p = \frac{1}{2}$, $p = \frac{3}{2}$, --- $p = \frac{q}{2}$; et in (298. § I.) pro $p = -\frac{q}{2}$ ponatur $m = 1$, $m = 2$, $m = 3$, --- $m = n - 1$.

$$\int \frac{dx}{x\sqrt{(\alpha+\beta x)^q}} = \frac{2}{(q-2)\alpha\sqrt{(\alpha+\beta x)^{q-2}}} + \frac{2}{(q-4)\alpha^2\sqrt{(\alpha+\beta x)^{q-4}}}$$
$$+ \frac{2}{(q-6)\alpha^3\sqrt{(\alpha+\beta x)^{q-6}}} + \cdots\cdots$$
$$+ \frac{2}{1.\alpha^{\frac{q-1}{2}}.\sqrt{(\alpha+\beta x)}} + \frac{1}{\alpha^{\frac{q-1}{2}}} \int \frac{dx}{x\sqrt{(\alpha+\beta x)}}.$$

$$\int \frac{dx}{x^n\sqrt{(\alpha+\beta x)^q}} = \Big(\frac{-1}{(n-1)\alpha x^{n-1}} + \frac{(q+2n-4)\beta}{2(n-1)(n-2)\alpha^2 . x^{n-2}}$$
$$- \frac{(q+2n-4)(q+2n-6)\beta^2}{2^2(n-1)(n-2)(n-3)\alpha^3.x^{n-3}} + \cdots\cdots$$
$$\mp \frac{(q+2n-4)(q+2n-6)\cdots(q+2)\beta^{n-2}}{2^{n-2}.(n-1)(n-2)\cdots 2.1.\alpha^{n-1}.x}\Big)\frac{1}{\sqrt{(\alpha+\beta x)^{q-2}}}$$
$$\mp \frac{(q+2n-4)(q+2n-6)\cdots(q+2)q\beta^{n-1}}{2^{n-1}.(n-1)(n-2)\cdots 2.1.\alpha^{n-1}} \int \frac{dx}{x\sqrt{(\alpha+\beta x)^q}}.$$

306. Problema.

Dato integrali $\int x^m \partial x(\alpha+\gamma x^2)^p$ *invenire integralia* $\int x^{m+2}\partial x(\alpha+\gamma x^2)^p$, $\int x^m \partial x(\alpha+\gamma x^2)^{p+1}$.

Solutio.

Primum invenies, si in (261. §. I.) ponas $\beta=0$, et $m+1$ loco m: alterum vero obtinebis ex (261. §. III.) posito $\beta=0$, et $m+1$ loco m.

$$\text{I. } \int x^{m+2}\partial x(\alpha+\gamma x^2)^p = \frac{x^{m+1}(\alpha+\gamma x^2)^{p+1}}{\gamma(m+2p+3)} - \frac{(m+1)\alpha}{\gamma(m+2p+3)}\int x^m \partial x(\alpha+\gamma x^2)^p.$$

$$\text{II. } \int x^m \partial x(\alpha+\gamma x^2)^{p+1} = \frac{x^{m+1}(\alpha+\gamma x^2)^{p+1}}{2p+m+3} + \frac{2\alpha(p+1)}{2p+m+3}\int x^m \partial x(\alpha+\gamma x^2)^p.$$

307. Corollarium 1.

Si in prima formula ponatur $-m$ loco m, et in secunda $-p$ loco p, prodibunt inde sequentes formulae.

$$\text{I. } \int\frac{\partial x(\alpha+\gamma x^2)^p}{x^m} = -\frac{(\alpha+\gamma x^2)^{p+1}}{(m-1)\alpha x^{m-1}} + \frac{\gamma(2p+3-m)}{(m-1)\alpha}\int\frac{\partial x(\alpha+\gamma x^2)^p}{x^{m-2}}.$$

$$\text{II. } \int\frac{x^m \partial x}{(\alpha+\gamma x^2)^p} = \frac{x^{m+1}}{2\alpha(p-1)(\alpha+\gamma x^2)^{p-1}} - \frac{(m+3-2p)}{2\alpha(p-1)}\int\frac{x^m \partial x}{(\alpha+\gamma x^2)^{p-1}}.$$

308. Corollarium 2.

Ex (306. §.) obtinentur sequentia integralia, per integrale $\int\partial x\sqrt{(\alpha+\gamma x^2)}$ ex (289. §.) notum determinata.

$$\int x^{2n}\partial x\sqrt{(\alpha+\gamma x^2)} = \Big(\frac{x^{2n-1}}{(2n+2)\gamma} - \frac{(2n-1)\alpha x^{2n-3}}{(2n+2)2n\gamma^2}$$
$$+ \frac{(2n-1)(2n-3)\alpha^2 . x^{2n-5}}{(2n+2)2n(2n-2)\gamma^3} - \cdots + \cdots$$
$$\pm \frac{(2n-1)(2n-3)\cdots 5.3.\alpha^{n-1}.x}{(2n+2)2n(2n-2)\cdots 6.4.\gamma^n}\Big)\sqrt{(\alpha+\gamma x^2)^3}$$
$$\mp \frac{(2n-1)(2n-3)\cdots 5.3.1.\alpha^n}{(2n+2)2n(2n-2)\cdots 6.4.\gamma^n}\int\partial x\sqrt{(\alpha+\gamma x^2)}.$$

$$\int x^{2n}\partial x\sqrt{(\alpha+\gamma x^2)^q} = \Big(\frac{\sqrt{(\alpha+\gamma x^2)^q}}{2n+q+1} + \frac{q\alpha\sqrt{(\alpha+\gamma x^2)^{q-2}}}{(2n+q+1)(2n+q-1)}$$
$$+ \frac{q(q-2)\alpha^2\sqrt{(\alpha+\gamma x^2)^{q-4}}}{(2n+q+1)(2n+q-1)(2n+q-3)} + \cdots$$
$$+ \frac{q(q-2)(q-4)\cdots 7.5.\alpha^{\frac{q-3}{2}}.\sqrt{(\alpha+\gamma x^2)^3}}{(2n+q+1)(2n+q-1)\cdots(2n+6)(2n+4)}\Big)x^{2n+1}$$

+

$$+\frac{q(q-2)(q-4)\ \text{---}\ 7.5.3.\alpha^{\frac{q-1}{2}}}{(2n+q+1)(2n+q-1)\ \text{---}\ (2n+6)(2n+4)}\int x^{2n}\varepsilon x\sqrt{(\alpha+\gamma x^2)}$$

309. Corollarium 3.

Ex (306. §. I.) et (307. §. II.) elicientur sequentia per integrale $\int\frac{\varepsilon x}{\sqrt{(\alpha+\gamma x^2)}}$ ex (287. §.) notum determinata integralia.

$$\int\frac{x^{2n}\varepsilon x}{\sqrt{(\alpha+\gamma x^2)}}=\Big(\frac{x^{2n-1}}{2n.\gamma}-\frac{(2n-1)\alpha x^{2n-3}}{2n(2n-2)\gamma^2}$$

$$+\frac{(2n-1)(2n-3)\alpha^2.x^{2n-5}}{2n(2n-2)(2n-4)\gamma^3}-\ \text{---}\ +\ \text{---}$$

$$\text{---}\ \pm\frac{(2n-1)(2n-3)\ \text{---}\ 5.3.\alpha^{n-1}.x}{2n(2-n2)\ \text{---}\ 6.4.2.\gamma^n}\Big)\sqrt{(\alpha+\gamma x^2)}$$

$$\mp\frac{(2n-1)(2n-3)\ \text{---}\ 5.3.1.\alpha^n}{2n(2n-2)\ \text{---}\ 6.4.2.\gamma^n}\int\frac{\varepsilon x}{\sqrt{(\alpha+\gamma x^2)}}.$$

$$\int\frac{x^{2n}\varepsilon x}{\sqrt{(\alpha+\gamma x^2)^q}}=\Big(\frac{1}{(q-2)\alpha\sqrt{(\alpha+\gamma x^2)^{q-2}}}-\frac{2n-q+3}{(q-2)(q-4)\alpha^2\sqrt{(\alpha+\gamma x^2)^{q-4}}}$$

$$+\frac{(2n-q+3)(2n-q+5)}{(q-2)(q-4)(q-6)\alpha^3\sqrt{(\alpha+\gamma x^2)^{q-6}}}-\ \text{---}\ +\ \text{---}$$

$$\text{---}\ \pm\frac{(2n-q+3)(2n-q+5)\ \text{---}\ (2n-2)}{(q-2)(q-4)\ \text{---}\ 5.3.1.\alpha^{\frac{q-1}{2}}\sqrt{(\alpha+\gamma x^2)}}\Big)x^{2n+1}$$

$$\mp\frac{(2n-q+3)(2n-q+5)\ \text{---}\ (2n-2)2n}{(q-2)(q-4)\ \text{---}\ 5.3.1,\alpha^{\frac{q-1}{2}}}\int\frac{x^{2n}\varepsilon x}{\sqrt{(\alpha+\gamma x^2)}}.$$

Hinc patet, secundum integrale debere fieri perfecte algebraicum, independens a primo, si est $q=2n+3$, aut $q>2n+3$.

310. Corollarium 4.

Porro ex (306. 307. §.) derivari possunt sequentia integralia, per integralia ex (291. 289. §.) nota perfecte determinata.

$$\int\frac{\varepsilon x\sqrt{(\alpha+\gamma x^2)^q}}{x}=\frac{\sqrt{(\alpha+\gamma x^2)^q}}{q}+\frac{\alpha\sqrt{(\alpha+\gamma x^2)^{q-2}}}{q-2}$$

$$+\frac{\alpha^2\sqrt{(\alpha+\gamma x^2)^{q-4}}}{q-4}+\ \text{---}\ +\ \text{---}$$

$$+\frac{\alpha^{\frac{q-1}{2}}.\sqrt{(\alpha+\gamma x^2)}}{1}+\alpha^{\frac{q+1}{2}}\int\frac{\varepsilon x}{x\sqrt{(\alpha+\gamma x^2)}}.$$

$\int\varepsilon x$

$$\int\frac{\partial x\sqrt{(\alpha+\gamma x^2)^q}}{x^{2n+1}}=-\Big(\frac{1}{2n.\alpha x^{2n}}+\frac{(q-2n+2)\gamma}{2n(2n-2)\alpha^2.x^{2n-2}}$$
$$+\frac{(q-2n+2)(q-2n+4)\gamma^2}{2n(2n-2)(2n-4)\alpha^3.x^{2n-4}}+\text{-----}$$
$$+\frac{(q-2n+2)(q-2n+4)\cdots(q-2)\gamma^{n-1}}{2n(2n-2)\text{ --- }4.2.\alpha^n.x^2}\Big)\sqrt{(\alpha+\gamma x^2)^{q+2}}$$
$$+\frac{(q-2n+2)(q-2n+4)\text{---}(q-2)q\gamma^n}{2n(2n-2)(2n-4)\text{---}4.2.\alpha^n}\int\frac{\partial x\sqrt{(\alpha+\gamma x^2)^q}}{x}.$$

$$\int\partial x\sqrt{(\alpha+\gamma x^2)^q}=\Big(\frac{\sqrt{(\alpha+\gamma x^2)^q}}{q+1}+\frac{q\alpha\sqrt{(\alpha+\gamma x^2)^{q-2}}}{(q+1)(q-1)}$$
$$+\frac{q(q-2)\alpha^2\sqrt{(\alpha+\gamma x^2)^{q-4}}}{(q+1)(q-1)(q-3)}+\text{-------}$$
$$+\frac{q(q-2)\text{ ---- }7.5.\alpha^{\frac{q-3}{2}}.\sqrt{(\alpha+\gamma x^2)^3}}{(q+1)(q-1)(q-3)\text{ --- }6.4}\Big)x$$
$$+\frac{q(q-2)(q-4)\text{ --- }7.5.3.\alpha^{\frac{q-1}{2}}}{(q+1)(q-1)(q-3)\text{ --- }6.4}\int\partial x\sqrt{(\alpha+\gamma x^2)}.$$

$$\int\frac{\partial x\sqrt{(\alpha+\gamma x^2)^q}}{x^{2n}}=-\Big(\frac{1}{(2n-1)\alpha x^{2n-1}}+\frac{(q-2n+3)\gamma}{(2n-1)(2n-3)\alpha^2.x^{2n-3}}$$
$$+\frac{(q-2n+3)(q-2n+5)\gamma^2}{(2n-1)(2n-3)(2n-5)\alpha^3.x^{2n-5}}+\text{-----}$$
$$+\frac{(q-2n+3)(q-2n+5)\text{--}(q-1)\gamma^{n-1}}{(2n-1)(2n-3)\text{ --- }3.1.\alpha^n.x}\Big)\sqrt{(\alpha+\gamma x^2)^{q+2}}$$
$$+\frac{(q-2n+3)(q-2n+5\text{--}(q-1)(q+1)\gamma^n}{(2n-1)(2n-3)\text{ --- }3.1.\alpha^n}\int\partial x\sqrt{(\alpha+\gamma x^2)^q}.$$

Erit autem ultimum integrale perfecte algebraicum, independens a tertio, si fuerit $2n=q+3$ vel $2n>q+3$.

311. Corollarium 5.

Denique ex (306. 307. §.) derivari possunt sequentia integralia, ad integrale $\int\frac{\partial x}{x\sqrt{(\alpha+\gamma x^2)}}$ ex (291. §.) notum relata.

$$\int\frac{\partial x}{x\sqrt{(\alpha+\gamma x^2)^q}}=\frac{1}{(q-2)\alpha\sqrt{(\alpha+\gamma x^2)^{q-2}}}+\frac{1}{(q-4)\alpha^2\sqrt{(\alpha+\gamma x^2)^{q-4}}}$$
$$+\frac{1}{(q-6)\alpha^3\sqrt{(\alpha+\gamma x^2)^{q-6}}}+\text{--------}$$

+

$$+\frac{-1}{1.\alpha^{\frac{q-1}{2}}.\sqrt{(\alpha+\gamma x^2)}}+\frac{1}{\alpha^{\frac{q-1}{2}}}\int\frac{\partial x}{x\sqrt{(\alpha+\gamma x^2)}}.$$

$$\int\frac{\partial x}{x^{2n+1}\sqrt{(\alpha+\gamma x^2)^q}}=\Big(\frac{-1}{2n.\alpha x^{2n}}+\frac{(q+2n-2)\gamma}{2n(2n-2)\alpha^2.x^{2n-2}}$$
$$-\frac{(q+2n-2)(q+2n-4)\gamma^2}{2n(2n-2)(2n-4)\alpha^3.x^{2n-4}}+\;----$$
$$\pm\frac{(q+2n-2)(q+2n-4)---(q+2)\gamma^{n-1}}{2n(2n-2)(2n-4)---6.4.2.\alpha^n.x^2}\Big)\frac{1}{\sqrt{(\alpha+\gamma x^2)^{q-2}}}$$
$$\pm\frac{(q+2n-2)(q+2n-4)---(q+2)q\gamma^n}{2n(2n-2)(2n-4)---6.4,2.\alpha^n}\int\frac{\partial x}{x\sqrt{(\alpha+\gamma x^2)^q}}.$$

312. Problema.

Integrare exponentem differentialem $x^m\partial x(a+bx^n)^p$, *in quo* p *est numerus integer negativus, alteruter vero, vel uterque exponens* m, n, *fractus ita tamen, ut* $\frac{m+1}{n}$ *aequetur alicui fractioni* $\frac{v}{2}$ *denominatoris* 2.

Solutio.

Denotante q numerum integrum positivum, ponatur $p=-q$; erit pro $z=a+bx^n$

$$x^m\partial x(a+bx^n)^p=\frac{x^m\partial x}{(a+bx^n)^q}=\frac{1}{nb^{\frac{m+1}{n}}}.\frac{\partial z(z-a)^{\frac{m+1}{n}-1}}{z^q}.$$

Quare, cum per hypothesin sit $\frac{m+1}{n}$, adeoque etiam $\frac{m+1}{n}-1$ certa fractio $\pm\frac{v}{2}$ denominatoris 2; poterit quivis hujuscemodi exponens differentialis per (304. 305. §.) perfecte integrari.

313. Problema.

Integrare exponentem irrationalem $x^m\partial x(a+bx^n)^{\frac{u}{2}}$, *in quo est* $\frac{m+1}{n}$ *aut numerus integer negativus, aut aliqua fractio, positiva vel negativa, denominatoris* 2, *quin ideo* $\frac{m+1}{n}+\frac{u}{2}$ *aequetur numero integro negativo* (283. §.).

Solutio.

Pro $z = \sqrt{(a+bx^n)}$ erit

$$x^m \partial x (a+bx^n)^{\frac{u}{2}} = \frac{2}{nb^{\frac{m+1}{n}}} . z^{u+1} \partial z (z^2 - a)^{\frac{m+1}{n} - 1}$$

Cum jam u ſit numerus integer, poſitivus vel negativus, et $\frac{m+1}{n}$, adeoque etiam $\frac{m+1}{n} - 1$ vel numerus integer negativus, vel aliqua fractio $\pm \frac{v}{2}$ denominatoris 2; poterit hic exponens omni caſu per (308. - - - 311. §.) perfecte integrari.

314. Problema.

Integrare exponentem differentialem $x^m \partial x (a+bx^n)^{\frac{u}{\mu}}$ *pro quacunque fractione* $\frac{u}{\mu}$ *denominatoris* $\mu > 2$, *modo ſit* $\frac{m+1}{n}$ *numerus integer negativus, aut* $\frac{m+1}{n} + \frac{u}{\mu}$ *numerus integer poſitivus.*

Solutio.

I. Pro $z = \sqrt[\mu]{(a+bx^n)}$ fiet

$$x^m \partial x (a+bx^n)^{\frac{u}{\mu}} \frac{\mu}{nb^{\frac{m+1}{n}}} . z^{u+\mu-1} . \partial z (z^\mu - a)^{\frac{m+1}{n} - 1}.$$

II. Pro $z = \sqrt[\mu]{\frac{a+bx^n}{x^n}}$ erit

$$x^m \partial x (a+bx^n)^{\frac{u}{\mu}} = - \frac{\mu a^{\frac{m+1}{n} + \frac{u}{\mu}}}{n} . \frac{\partial z \, z^{u+\mu-1}}{(z^\mu - b)^{\frac{m+1}{n} + \frac{u}{\mu} + 1}}.$$

Iam vero u, μ ſunt numeri integri per hypotheſin: ſi ergo ſit $\frac{m+1}{n}$ numerus integer negativus; poterit integratio, adhibita prima transformatione, per (285. §.) perfici: ſi autem ſit $\frac{m+1}{n} + \frac{u}{\mu}$ numerus integer poſitivus; poterit ea per (285. §.) perfici, adhibita ſecunda transformatione.

315. Problema.

Definire conditiones, quae exponentem differentialem $\varepsilon y = x^m \varepsilon x (a + b x^n)^p$ *reddant perfecte integrabilem.*

Solutio.

1. Omni casu, seu sit εy functio rationalis, seu alia quaecunque functio, modo sit $\frac{m+1}{n}$ numerus integer positivus, vel $\frac{m+1}{n} + p$ numerus integer negativus, poterit datus exponens per praecepta 4ti capitis perfecte integrari.

2. Si nulla harum conditionum (1) adsit, attendatur ad exponentem p. Existente p numero integro positivo, poterit integrale completum per (250. §.) determinari: si autem sit p numerus integer negativus, unico casu poterit datus exponens integrari, nimirum per (312. §.), si fuerit $\frac{m+1}{n}$ aliqua fractio denominatoris 2.

3. Si vero, absentibus his omnibus conditionibus (1) (2), sit p numerus fractus $\frac{u}{\mu}$ denominatoris $\mu = 2$, vel $\mu > 2$: casu primo aut pertinebit datus exponens differentialis ad formulas, quarum integralia in (299 --- 311. §.) sunt exposita, aut tunc solum erit is perfecte integrabilis, si fuerit $\frac{m+1}{n}$ vel numerus integer negativus, vel aliqua fractio, positiva aut negativa, denominatoris 2. (313. §.): casu autem altero non poterit datus exponens integrari, nisi per (314. §.), si sit $\frac{m+1}{n}$ numerus integer negativus, vel $\frac{m+1}{n} + \frac{u}{\mu}$ numerus integer positivus.

316. Corollarium.

Si fuerit integrandus exponens differentialis formae $\varepsilon y = x^r \varepsilon x (a x^u + b x^v)^k$; fiat $\varepsilon y = x^{r+uk} . \varepsilon x (a + b x^{v-u})^k$, vel $\varepsilon y = x^{r+vk} . \varepsilon x (a x^{u-v} + b)^k$, tum investigetur per (315. §.), utrum, et qua methodo is sit perfecte integrabilis.

Scholion.

Omnes exponentes differentiales, quorum integrationem in praesenti capite exposuimus, ita sunt comparati, ut illorum integratio ad pauca

praecepta generalia possit revocari: omnes sane, ut videbimus in sequenti capite, formam exponentium rationalium possunt induere, eoque ipso reddi integrabiles. Verum, cum haec integrandi methodus in adplicatione calculi integralis sit saepenumero admodum molesta; cumque hi ipsi exponentes irrationales frequentissime ingrediantur in disquisitiones analyticas; et demum innumeri alii exponentes irrationales ad illos summo calculi compendio possint revocari: operae pretium erat, completa illorum integralia diligentius, quam solet fieri, evolvere, aliamque simpliciorem methodum integrandi ei, quae pro trinomiis irrationalibus a doctoribus calculi integralis passim praecipitur, substituere. Artificia, quibus exponentes irrationales possunt reddi integrabiles, quin transformatione in rationales opus sit, usu et assidua exercitatione optime discuntur; praecipua tamen ad bina sequentia capita possunt revocari.

I. Multi exponentes irrationales, qui non videntur esse integrabiles, sola multiplicatione, divisione, aut radicis extractione redduntur integrabiles: quaenam harum operationum institui debeat, ex comparatione formae, sub qua exponens integrandus proponitur, cum formis illorum exponentium, quorum integratio nota est, oportebit determinare.

II. Palmare artificium, quo etiam in antecedentibus saepius jam usi sumus, consistit in introductione novae variabilis, quae, pro diversitate exponentium integrandorum, jam toti functioni signo radicali substanti, jam certae ejus parti, aut alteri cuipiam functioni solet aequari. Ceterum, denuo moneo tyrones, ut, dum formam exponentis integrandi attente expendunt, formas quoque exponentium differentialium, de quorum integratione superius est actum, praeceptaque calculi differentialis perpetuo prae oculis habeant. Sub his conditionibus eveniet sane saepenumero, ut certae functiones pro variabilibus sponte se offerant.

Scholion 2.

Quodsi autem nullum artificium in potestate habeatur, quo datus exponens irrationalis ita possit transformari, ut secundum praecedentia principia fiat, retenta irrationalitate, integrabilis; tentetur transformatio per sublationem irrationalitatis. Multi profecto occurrent exponentes, qui, licet ii nullo modo integrabiles esse videantur, si per substitutionem novae alicujus variabilis ab omni irrationalitate liberentur, fiant eo ipso perfecte

fecte integrabiles. Conſtat praeterea, quemvis poſſibilem exponentem differentialem formae rationalis per praecepta 4ti capitis perfecte poſſe integrari, modo concedatur reſolutio cujuslibet functionis integrae et rationalis in ſuos factores ſimplices: eo circa optandum eſſet, ut extaret aliqua generalis methodus omnibus exponentibus irrationalibus formam rationalium tribuendi, quam tamen penitus ignoramus. Quamobrem expendemus in ſequenti capite peculiares aliquot, latiſſime patentes, formas exponentium irrationalium, artificiaque docebimus, quibus ii ab irrationalitate liberari poſſunt: tum ſubjungemus methodum in deſperatis caſibus, quales in adplicatione calculi integralis fere perpetuo ſe offerunt, adproximandi ad integralia exponentium, quorum perfecta integratio nulla via poteſt impetrari.

CAPUT VI.

DE INTEGRATIONE EXPONENTIUM TRANSCENDENTIUM; TRANSFORMATIONE EXPONENTIUM IRRATIONALIUM IN RATIONALES; INTEGRATIONEQUE PER SERIES.

317. Theorema.

Assumtis signis $\int Y \varepsilon x$; ${}^2\!\int Y\varepsilon x = \int \varepsilon x \int Y \varepsilon x$; ${}^3\!\int Y \varepsilon x = \int \varepsilon x\, {}^2\!\int Y \varepsilon x$; *et generatim* ${}^n\!\int Y \varepsilon x = \int \varepsilon x\, {}^{n-1}\!\int Y \varepsilon x$; *erit pro duabus quibuscunque functionibus* Z, Y *variabilis absolutae* x.

$$\int Z Y \varepsilon x =$$

$$= Z\int Y\varepsilon x - \frac{\varepsilon Z}{\varepsilon x}\,{}^2\!\int Y\varepsilon x + \frac{\overset{2}{\varepsilon} Z}{\varepsilon x^2}\,{}^3\!\int Y\varepsilon x - \frac{\overset{3}{\varepsilon} Z}{\varepsilon x^3}\,{}^4\!\int Y\varepsilon x + \frac{\overset{4}{\varepsilon} Z}{\varepsilon x^4}\,{}^5\!\int Y\varepsilon x - \text{etc.}$$

Demonstratio.

1. Pro quolibet indice n integralis ${}^n\!\int Y\varepsilon x$ erit per (257. §.).

$$\pm\int\frac{\overset{n}{\varepsilon} Z}{\varepsilon x^{n-1}}\,{}^n\!\int Y\varepsilon x = \pm\frac{\overset{n}{\varepsilon} Z}{\varepsilon x^n}\int\varepsilon x\,{}^n\!\int Y\varepsilon x \mp \int\varepsilon\frac{\overset{n}{\varepsilon} Z}{\varepsilon x^n}\int\varepsilon x\,{}^n\!\int Y\varepsilon x =$$

$$= \pm\frac{\overset{n}{\varepsilon} Z}{\varepsilon x^n}\,{}^{n+1}\!\int Y\varepsilon x \mp \int\frac{\overset{n+1}{\varepsilon} Z}{\varepsilon x^n}\,{}^{n+1}\!\int Y\varepsilon x.$$

2. Cum igitur sit $\int Z Y\varepsilon x = Z\int Y\varepsilon x - \int\varepsilon Z\int Y\varepsilon x$ (257. §.); erit per (1) pro signis in theoremate assumtis

$$\int Z Y\varepsilon x = Z\int Y\varepsilon x - \frac{\varepsilon Z}{\varepsilon x}\,{}^2\!\int Y\varepsilon x + \int\frac{\overset{2}{\varepsilon} Z}{\varepsilon x}\,{}^2\!\int Y\varepsilon x =$$

$$= Z\int Y\varepsilon x - \frac{\varepsilon Z}{\varepsilon x}\,{}^2\!\int Y\varepsilon x + \frac{\overset{2}{\varepsilon} Z}{\varepsilon x^2}\,{}^3\!\int Y\varepsilon x - \int\frac{\overset{3}{\varepsilon} Z}{\varepsilon x^2}\,{}^3\!\int Y\varepsilon x =$$

$$= Z\int Y\varepsilon x - \frac{\varepsilon Z}{\varepsilon x}\,{}^2\!\int Y\varepsilon x + \frac{\overset{2}{\varepsilon} Z}{\varepsilon x^2}\,{}^3\!\int Y\varepsilon x - \frac{\overset{3}{\varepsilon} Z}{\varepsilon x^3}\,{}^4\!\int Y\varepsilon x + \int\frac{\overset{4}{\varepsilon} Z}{\varepsilon x^3}\,{}^4\!\int Y\varepsilon x,$$ et sic porro.

318. Co-

318. Corollarium 1.

Pro $Z = y$ et $Y = x^0$ obtinebitur per (317. §.) sequens notissima series *Bernoulliana*, cujus ope integrale cujusvis exponentis differentialis $y \epsilon x$ per exponentes ϵy, $\overset{2}{\epsilon} y$, $\overset{3}{\epsilon} y$, etc. rationum differentialium functionis y commode posset determinari, nisi ea id vitii haberet, quod, si y non sit functio integra rationalis, in infinitum excurrat.

$$\int y \epsilon x = xy - \frac{x^2 \epsilon y}{2 . \epsilon x} + \frac{x^3 . \overset{2}{\epsilon} y}{2 . 3 . \epsilon x^2} - \frac{x^4 . \overset{3}{\epsilon} y}{2 . 3 . 4 . \epsilon x^3} + \text{etc.}$$

319. Corollarium 2.

Pro $\varphi = k^x$ est $\epsilon \varphi = k^x \epsilon x . lk$ (112. §.): igitur debet esse $\int k^x \epsilon x = \frac{k^x}{la}$ (239. 241. §.). Quamobrem, posito $Y = a^x$, erit in (317. §.) $\int Y \epsilon x = \frac{a^x}{la}$; $\overset{2}{\int} Y \epsilon x = \frac{a^x}{(la)^2}$; $\overset{3}{\int} Y \epsilon x = \frac{a^x}{(la)^3}$; et sic porro. Pro qualibet functione Z variabilis absolutae x erit ergo per (317. §.).

$$\int Z a^x \epsilon x = \frac{Z a^x}{la} - \frac{a^x \epsilon Z}{(la)^2 \epsilon x} + \frac{a^x \overset{2}{\epsilon} Z}{(la)^3 \epsilon x^2} - \frac{a^x \overset{3}{\epsilon} Z}{(la)^4 \epsilon x^3} + \text{etc.}$$

320. Corollarium 3.

Series haec (319. §.) abrumpetur, si aliquis exponentium differentialium ϵZ, $\overset{2}{\epsilon} Z$, $\overset{3}{\epsilon} Z$, $\overset{4}{\epsilon} Z$, etc. fuerit aequalis nihilo, quo casu dabit illa integrale completum exponentis differentialis $Z a^x \epsilon x$: exponens hic erit ergo per (319. §.) perfecte integrabilis, quoties fuerit Z aliqua functio integra rationalis variabilis absolutae x.

321. Corollarium 4.

Pro $Z = x^n$ obtinebitur ex (320. §.) sequens series, quae, nisi sit n numerus integer positivus, in infinitum excurret, abrumpetur autem, completumque dabit integrale exponentis differentialis $a^x x^n \epsilon x$, quoties fuerit n numerus integer positivus.

$$\int a^x x^n \epsilon x = a^x \Big(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{- - - - - -} \Big) + C.$$

322. Co-

322. Corollarium 5.

Exponens negativus $n = -1$ dabit in (321. §.) sequens integrale, pro quo finita expressio ignoratur.

$$\int \frac{a^x \partial x}{x} = a^x \left(\frac{1}{x\,la} + \frac{1}{x^2 (la)^2} + \frac{1.2}{x^3 (la)^3} + \frac{1.2.3}{x^4 (la)^4} + \text{etc.} \right) + C.$$

323. Corollarium 6.

Cum ubique logarithmi naturales subintelligantur, abibunt praecedentes series (319. 321. 322. §.), pro basi $a = e$ logarithmorum naturalium, in sequentes.

$$\int Z e^x \partial x = Z e^x - \frac{e^x \partial Z}{\partial x} + \frac{e^x \overset{2}{\partial} Z}{\partial x^2} - \frac{e^x \overset{3}{\partial} Z}{\partial x^3} + \text{etc.}$$

$$\int x^n e^x \partial x = e^x (x^n - n x^{n-1} + n(n-1) x^{n-2} - n(n-1)(n-2) x^{n-3} + \text{etc.}) + C.$$

$$\int \frac{e^x \partial x}{x} = e^x \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1.2}{x^3} + \frac{1.2.3}{x^4} + \frac{1.2.3.4}{x^5} + \text{etc.} \right) + C.$$

324. Theorema.

Assumtis signis $\int Y \partial x$; $\overset{I}{\int} Y \partial x = \int \frac{\partial x}{x} \int Y \partial x$; $\overset{II}{\int} Y \partial x = \int \frac{\partial x}{x} \overset{I}{\int} Y \partial x$, $\overset{III}{\int} Y \partial x = \int \frac{\partial x}{x} \overset{II}{\int} Y \partial x$, *et ita porro; item* $\partial Z = A \partial x$; $\partial . A x = B \partial x$; $\partial . B x = C \partial x$, *et sic porro: erit pro quibuslibet duabus functionibus* Z, Y *variabilis absolutae* x.

$$\int Z Y \partial x =$$
$$= Z \int Y \partial x - A x \overset{I}{\int} Y \partial x + B x \overset{II}{\int} Y \partial x - C x \overset{III}{\int} Y \partial x + \text{etc.}$$

Demonstratio.

1. Per (257. §.) erit generatim pro quocunque indice N integralis $\overset{N}{\int} Y \partial x$.

$$\pm \int M \partial x \overset{N}{\int} Y \partial x = \pm \int \frac{M x \partial x}{x} \overset{N}{\int} Y \partial x =$$
$$= \pm M x \int \frac{\partial x}{x} \overset{N}{\int} Y \partial x \mp \int \partial . M x \int \frac{\partial x}{x} \overset{N}{\int} Y \partial x =$$
$$= \pm M x . \overset{N+1}{\int} Y \partial x \mp \int \partial . M x . \overset{N+1}{\int} Y \partial x =$$
$$= \pm M x . \overset{N+1}{\int} Y \partial x \mp \int P \partial x \overset{N+1}{\int} Y \partial x$$

posito $\partial . M x = P \partial x$.

2. Qua-

2. Quare, cum sit $\int ZY\partial x = Z\int Y\partial x - \int \partial Z \int Y\partial x$ (257. §.) $= Z\int Y\partial x - \int A\partial x \int Y\partial x$; erit quoque pro signis in theoremate assumtis per (1)

$$\int ZY\partial x = Z\int Y\partial x - Ax\,{}^{\mathrm{I}}\!\int Y\partial x + \int B\partial x\,.\,{}^{\mathrm{I}}\!\int Y\partial x =$$
$$= Z\int Y\partial x - Ax\,{}^{\mathrm{I}}\!\int Y\partial x + Bx\,.\,{}^{\mathrm{II}}\!\int Y\partial x - \int C\partial x\,{}^{\mathrm{II}}\!\int Y\partial x =$$
$$= Z\int Y\partial x - Ax\,{}^{\mathrm{I}}\!\int Y\partial x + Bx\,{}^{\mathrm{II}}\!\int Y\partial x - Cx\,{}^{\mathrm{III}}\!\int Y\partial x + \int D\partial x\,{}^{\mathrm{III}}\!\int Y\partial x =$$
$$= Z\int Y\partial x - Ax\,{}^{\mathrm{I}}\!\int Y\partial x + Bx\,{}^{\mathrm{II}}\!\int Y\partial x - Cx\,{}^{\mathrm{III}}\!\int Y\partial x + \text{etc.}$$

325. Corollarium 1.

Pro $Z = (lx)^n$ erit in (324. §.) $\partial Z = n(lx)^{n-1}.\frac{\partial x}{x}$; $Ax = n(lx)^{n-1}$; $\partial.Ax = n(n-1)(lx)^{n-2}.\frac{\partial x}{x} = B\partial x$; $\partial.Bx = n(n-1)(n-2)(lx)^{n-3}.\frac{\partial x}{x}$; et sic porro per (104. 110. §.): pro his valoribus in (324. §.) substitutis obtinebitur sequens formula

$$\int Y(lx)^n\partial x = (lx)^n\int Y\partial x - n(lx)^{n-1}.\,{}^{\mathrm{I}}\!\int Y\partial x$$
$$+ n(n-1)(lx)^{n-2}.\,{}^{\mathrm{II}}\!\int Y\partial x - n(n-1)(n-2)(lx)^{n-3}.\,{}^{\mathrm{III}}\!\int Y\partial x$$
$$+ n(n-1)(n-2)(n-3)(lx)^{n-4}.\,{}^{\mathrm{IV}}\!\int Y\partial x - \text{etc.}$$

Quodsi ergo fuerit n numerus integer positivus, abrumpetur haec series: et ideo, si praeterea exponentes differentiales $\frac{\partial x}{x}\int Y\partial x$, $\frac{\partial x}{x}\,{}^{\mathrm{I}}\!\int Y\partial x$, $\frac{\partial x}{x}\,{}^{\mathrm{II}}\!\int Y\partial x$, $\frac{\partial x}{x}\,{}^{\mathrm{III}}\!\int Y\partial x$, etc. (324. §.) fuerint perfecte integrabiles, dabit illa completum integrale exponentis differentialis $Y(lx)^n\partial x$.

326. Corollarium 2.

Sit igitur $Y = x^m$; fiet $\int Y\partial x = \frac{x^{m+1}}{m+1}$; ${}^{\mathrm{I}}\!\int Y\partial x = \frac{x^{m+1}}{(m+1)^2}$; ${}^{\mathrm{II}}\!\int Y\partial x = \frac{x^{m+1}}{(m+1)^3}$; et ita porro: ob (325. §.) obtinebimus pro his valoribus sequens integrale, constans terminis numero finitis, si n est numerus integer positivus.

$$\int x^m(lx)^n\partial x = \left(\frac{(lx)^n}{m+1} - \frac{n(lx)^{n-1}}{(m+1)^2} + \frac{n(n-1)(lx)^{n-2}}{(m+1)^3}\right.$$
$$\left. - \frac{n(n-1)(n-2)(lx)^{n-3}}{(m+1)^4} + \text{etc.}\right)x^{m+1} + C.$$

327. Corollarium 3.

Series haec (326. §.) valorem infinitum obtinebit, si fiat $m = -1$, deberet autem illa dare integrale $\int \frac{(lx)^n}{x}\partial x$. Verum constat esse $\partial . \frac{(lx)^{n+1}}{n+1} = \frac{(lx)^n}{x}\partial x$ (104. 110. §.): debet ergo esse per (239. §.).

$$\int \frac{(lx)^n}{x}\partial x = \frac{(lx)^{n+1}}{n+1} + C.$$

328. Corollarium 4.

Exponens $n = -1$ in (326. §.) dabit sequens integrale.

$$\int \frac{x^m \partial x}{lx} = \left(\frac{1}{(m+1)lx} + \frac{1}{(m+1)^2(lx)^2} + \frac{1.2}{(m+1)^3(lx)^3} + \frac{1.2.3}{(m+1)^4(lx)^4} + \frac{1.2.3.4}{(m+1)^5(lx)^5} + \text{etc.} \right) x^{m+1} + C.$$

Hinc pro $m = 0$ fiet.

$$\int \frac{\partial x}{lx} = \left(\frac{1}{lx} + \frac{1}{(lx)^2} + \frac{1.2}{(lx)^3} + \frac{1.2.3}{(lx)^4} + \frac{1.2.3.4}{(lx)^5} + \text{etc.} \right) x + C.$$

329. Corollarium 5.

Pro $Z = x^{m+1}$ et $Y = \frac{1}{x(lx)^n}$ invenies per (324. §.) sequentem formulam integralem.

$$\int \frac{x^m \partial x}{(lx)^n} = -\left(\frac{1}{(n-1)(lx)^{n-1}} + \frac{m+1}{(n-1)(n-2)(lx)^{n-2}} + \frac{(m+1)^2}{(n-1)(n-2)(n-3)(lx)^{n-3}} + \frac{(m+1)^3}{(n-1)(n-2)(n-3)(n-4)(lx)^{n-4}} + \text{etc.} \right) x^{m+1} + \frac{(m+1)^{n-1}}{(n-1)(n-2)(n-3) - - - 3.2.1} \int \frac{x^m \partial x}{lx}.$$

Verum ignoramus artificium, quo exponens differentialis $\frac{x^m \partial x}{lx}$ ad perfectam integrationem recipiendam disponi queat.

330. Problema.

Integrare exponentem differentialem $\frac{\partial \Phi \, \text{Sin}\, \Phi^m}{(a + b\, \text{Cos}\, \Phi)^p}$.

Solutio.

Ob $\partial\varphi = -\frac{\partial \operatorname{Cos}\varphi}{\operatorname{Sin}\varphi}$ (119. §.) erit

$$\frac{\partial\varphi \operatorname{Sin}\varphi^m}{(a+b\operatorname{Cos}\varphi)^n} = -\frac{\operatorname{Sin}\varphi^{m-1}.\partial \operatorname{Cos}\varphi}{(a+b\operatorname{Cos}\varphi)^n}.$$

Pro $a+b\operatorname{Cos}\varphi = x$ fiet

$$\operatorname{Cos}\varphi = \frac{x-a}{b};\ \operatorname{Sin}\varphi = \frac{1}{b}\sqrt{(b^2-a^2+2ax-x^2)};\ \partial\operatorname{Cos}\varphi = \frac{\partial x}{b}:$$

pro his valoribus transformabitur datus exponens in

$$\frac{\partial\varphi \operatorname{Sin}\varphi^m}{(a+b\operatorname{Cos}\varphi)^n} = -\frac{1}{b^m}\cdot\frac{\partial x\sqrt{(b^2-a^2+2ax-x^2)^{m-1}}}{x^n}.$$

Hinc evidenter patet, omnem ejusmodi exponentem differentialem per praecepta 4ti, aut 5ti capitis perfecte esse integrabilem, modo sint m, n numeri integri.

331. Corollarium 1.

Pro $a=b$ in (330. §.) habebitur sequens formula differentialis

$$x = a + a\operatorname{Cos}\varphi = a(1+\operatorname{Cos}\varphi)$$

$$\frac{\partial\varphi \operatorname{Sin}\varphi^m}{(a+a\operatorname{Cos}\varphi)^n} = \frac{\partial\varphi \operatorname{Sin}\varphi^m}{a^n(1+\operatorname{Cos}\varphi)^n} = -\frac{1}{a^m}\cdot x^{\frac{m-2n-1}{2}}.\partial x\sqrt{(2a-x)^{m-1}}$$

Quare erunt omnes exponentes differentiales hujus formae per praecepta 4ti aut 5ti capitis perfecte integrabiles, si pro numero integro m fuerit n numerus integer, aut fractus denominatoris 2.

332. Corollarium 2.

Pro $m=0$ et $n=1$ in (330. §.) determinabis per (230. §.) sequentia integralia (I) (II): et si praeterea sit $a=b$, obtinebis sequens integrale (III).

Si est $a>b$

$$\text{I. } \int\frac{\partial\varphi}{a+b\operatorname{Cos}\varphi} = \frac{1}{\sqrt{(a^2-b^2)}}\operatorname{Arc}\operatorname{Cos}\frac{b+a\operatorname{Cos}\varphi}{b\operatorname{Cos}\varphi+a}+C.$$

Si est $a<b$.

$$\text{II. } \int\frac{\partial\varphi}{a+b\operatorname{Cos}\varphi} = C+\frac{1}{\sqrt{(b^2-a^2)}}\,l\left(\frac{b+a\operatorname{Cos}\varphi+\operatorname{Sin}\varphi.\sqrt{(b^2-a^2)}}{a+b\operatorname{Cos}\varphi}\right).$$

Si est $a=b$.

$$\text{III. } \int\frac{\partial\varphi}{a+a\operatorname{Cos}\varphi} = \frac{\sqrt{(1-\operatorname{Cos}\varphi)}}{a\sqrt{(1+\operatorname{Cos}\varphi)}} = \frac{1}{a}\operatorname{Tang}\tfrac{1}{2}\varphi+C.$$

Eo autem determinato casu, quo fuerit $m = n = 1$ in (330. §.), fiet

$$\text{IV.} \int \frac{\partial\varphi \operatorname{Sin}\varphi}{a + b \operatorname{Cos}\varphi} = C - \frac{1}{b}\, l(a + b \operatorname{Cos}\varphi).$$

333. Corollarium 3.

Et pro $m=0$, $n=2$ in (330. §.) elicies inde per (290. 294. §. IV.) sequentia notatu digna integralia.

Si est $a > b$.

$$\text{I.} \int \frac{\partial\varphi}{(a + b \operatorname{Cos}\varphi)^2} =$$

$$= C - \frac{b \operatorname{Sin}\varphi}{(a^2 - b^2)(a + b \operatorname{Cos}\varphi)} + \frac{a}{\sqrt{(a^2 - b^2)^3}} \operatorname{Arc} \operatorname{Cos} \frac{b + a \operatorname{Cos}\varphi}{a + b \operatorname{Cos}\varphi}.$$

Si est $a < b$

$$\text{II.} \int \frac{\partial\varphi}{(a + b \operatorname{Cos}\varphi)^2} =$$

$$= C + \frac{b \operatorname{Sin}\varphi}{(b^2 - a^2)(a + b \operatorname{Cos}\varphi)} - \frac{a}{\sqrt{(b^2 - a^2)^3}}\, l\left(\frac{b + a \operatorname{Cos}\varphi + \operatorname{Sin}\varphi . \sqrt{(b^2 - a^2)}}{a + b \operatorname{Cosin}\varphi}\right).$$

334. Corollarium 5.

Eadem prorsus operatione, qua in resolutione problematis praecedentis sumus usi, obtinebitur sequens transformatio.

$$x = a + b \operatorname{Cos}\varphi.$$

$$\frac{\partial\varphi \operatorname{Cos}\varphi^m}{(a + b \operatorname{Cos}\varphi)^n} = -\frac{1}{b^m} \cdot \frac{(x - a)^m \partial x}{x^n \sqrt{(b^2 - a^2 + 2ax - x^2)}}.$$

Atque hinc fit perspicuum, exponentes differentiales hujus formae secundum principia quinti capitis perfecte integrabiles futuros, si m sit numerus integer positivus, et n numerus quiscunque integer, positivus aut negativus.

335. Corollarium. 6.

Abibit autem haec formula (334. §.) pro $a = b$ in sequentem.

$$x = a + a \operatorname{Cos}\varphi = a(1 - \operatorname{Cos}\varphi).$$

$$\frac{\partial\varphi \operatorname{Cos}\varphi^m}{(a + a \operatorname{Cos}\varphi)^n} = -\frac{1}{a^m} \cdot \frac{(x - a)^m \partial x}{x^{\frac{2n+1}{2}} \sqrt{(2a - x)}}.$$

Unde patet, pro quibus numeris m, n, quave determinata methodo exponens hic differentialis perfecte est integrabilis: nimirum per praecepta quarti vel quinti capitis, si m est numerus integer positivus, et n aut numerus

merus integer, positivus vel negativus, aut aequalis alicui fractioni $\pm\frac{v}{2}$, quo $2n$ fiat numerus integer: quodsi autem sit $m=\pm\frac{1}{2}$, aut quaecunque alia ejusmodi, positiva tamen, fractio $m=\frac{2r-1}{2}$, cum hoc casu simus habituri

$$\frac{(x-a)^{\frac{1}{2}}}{\sqrt{(2a-x)}}=\frac{x-a}{\sqrt{(-2a^2+3ax-x^2)}};$$

$$\text{aut}\quad\frac{(x-a)^{-\frac{1}{2}}}{\sqrt{(2a-x)}}=\frac{1}{\sqrt{(-2a^2+3ax-x^2)}};$$

$$\text{vel}\quad\frac{(x-a)^{\frac{2r-1}{2}}}{\sqrt{(2a-x)}}=\frac{(x-a)^r}{\sqrt{(-2a^2+3ax-x^2)}};$$

erit datus exponens per theoriam trinomiorum quinti capitis perfecte integrabilis, modo sit n aliqua fractio $\pm\frac{v}{2}$, ut $\frac{2n+1}{2}=\frac{\pm v+1}{2}$ sit numerus integer.

336. Corollarium 7.

Similes expressiones algebraicae invenientur pro exponentibus differentialibus $\frac{\partial\phi\,\mathrm{Cos}\,\phi^m}{(a+b\,\mathrm{Sin}\,\phi)^n}$, $\frac{\partial\phi\,\mathrm{Sin}\,\phi^m}{(a+b\,\mathrm{Sin}\,\phi)^n}$: si nimirum fiat $\partial\phi=\frac{\partial\,\mathrm{Sin}\,\phi}{\mathrm{Cos}\,\phi}$, tum

$$a+b\,\mathrm{Sin}\,\phi=x;$$

$$\text{erit I.}\quad\frac{\partial\phi\,\mathrm{Cos}\,\phi^m}{(a+b\,\mathrm{Sin}\,\phi)^n}=\frac{1}{b^m}\cdot\frac{\partial x\sqrt{(b^2-a^2+2ax-x^2)^{m-1}}}{x^n}.$$

$$\text{II.}\quad\frac{\partial\phi\,\mathrm{Sin}\,\phi^m}{(a+b\,\mathrm{Sin}\,\phi)^n}=\frac{1}{b^m}\cdot\frac{(x-a)^m\partial x}{x^n\sqrt{(b^2-a^2+2ax-x^2)}}.$$

Adeoque etiam posito $a=b$.

$$a+a\,\mathrm{Sin}\,\phi=x.$$

$$\text{III.}\quad\frac{\partial\phi\,\mathrm{Cos}\,\phi^m}{(a+a\,\mathrm{Sin}\,\phi)^n}=\frac{1}{a^m}\cdot x^{\frac{m-2n-1}{2}}\cdot\partial x\sqrt{(2a-x)^{m-1}}.$$

$$\text{IV.}\int\frac{\partial\,\mathrm{Sin}\,\phi^m}{(a+a\,\mathrm{Sin}\,\phi)^n}=\frac{1}{a^m}\cdot\frac{(x-a)^m\partial x}{x^{\frac{2n+1}{2}}\sqrt{(2a-x)}}.$$

Quapropter pendebit integratio horum exponentium differentialium a theoria duorum capitum praecedentium.

337. Corollarium 8.

Pro $a=0$, et $b=1$ derivabis ex (336. §. I.) sequentes exponentes differentiales, prout nimirum utrumque exponentem, m, n, aut alterutrum duntaxat, cum proprio, vel contrario ($-$) signo sumseris, vel etiam $m=0$, aut $n=0$ posueris. Patet autem, omnes hos exponentes, hac ratione transformatos, secundum praecedentia principia, ea praecipue, quae in capite quinto sunt exposita, commodissime posse integrari.

$$x=\mathrm{Sin}\,\varphi;\ \sqrt{(1-x^2)}=\mathrm{Cos}\,\varphi.$$

$$\text{I. } \int\frac{\partial\varphi\,\mathrm{Cos}\,\varphi^m}{\mathrm{Sin}\,\varphi^n}=\frac{\partial x\sqrt{(1-x^2)}^{\,m-1}}{x^n}.$$

$$\text{II. } \int\frac{\partial\varphi\,\mathrm{Sin}\,\varphi^n}{\mathrm{Cos}\,\varphi^m}=\frac{x^n\partial x}{\sqrt{(1-x^2)}^{\,m+1}}.$$

$$\text{III. } \int\frac{\partial\varphi}{\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m}=\frac{\partial x}{x^n\sqrt{(1-x^2)}^{\,m+1}}.$$

$$\text{IV. } \int\partial\varphi\,\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m=x^n\partial x\sqrt{(1-x^2)}^{\,m-1}$$

$$\text{V. } \int\frac{\partial\varphi}{\mathrm{Sin}\,\varphi^n}=\frac{\partial x}{x^n\sqrt{(1-x^2)}}.$$

$$\text{VI. } \int\frac{\partial\varphi}{\mathrm{Cos}\,\varphi^m}=\frac{\partial x}{\sqrt{(1-x^2)}^{\,m+1}}.$$

$$\text{VII. } \int\partial\varphi\,\mathrm{Sin}\,\varphi^n=\frac{x^n\partial x}{\sqrt{(1-x^2)}}.$$

$$\text{VIII. } \int\partial\varphi\,\mathrm{Cos}\,\varphi^m=\partial x\sqrt{(1-x^2)}^{\,m-1}$$

338. Problema.

Integrare exponentes differentiales formulis generalibus $\frac{\partial\varphi\,\mathrm{Cot}\,\varphi^m}{(a+b\,\mathrm{Tang}\,\varphi)^n}$, $\frac{\partial\varphi\,\mathrm{Tang}\,\varphi^m}{(a+b\,\mathrm{Tang}\,\varphi)^n}$ *comprehensos.*

Solutio.

Cum sit $\partial\varphi=\frac{\partial\,\mathrm{Tang}\,\varphi}{\mathrm{Sec}\,\varphi^2}$ (119. §.); erit

$$\frac{\partial\varphi\,\mathrm{Cot}\,\varphi^m}{(a+b\,\mathrm{Tang}\,\varphi)^n}=\frac{\mathrm{Cot}\,\varphi^m.\,\partial\,\mathrm{Tang}\,\varphi}{\mathrm{Sec}\,\varphi^2\,(a+b\,\mathrm{Tang}\,\varphi)^n}$$

$$\frac{\partial\varphi\,\mathrm{Tang}\,\varphi^m}{(a+b\,\mathrm{Tang}\,\varphi)^n}=\frac{\mathrm{Tang}\,\varphi^m.\,\partial\,\mathrm{Tang}\,\varphi}{\mathrm{Sec}\,\varphi^2\,(a+b\,\mathrm{Tang}\,\varphi)^n}.$$

Pro

Pro $x = a + b \operatorname{Tang}\varphi$, hinc

$$\operatorname{Tang}\varphi = \frac{x-a}{b};\ \operatorname{Cot}\varphi = \frac{b}{x-a};\ \partial \operatorname{Tang}\varphi = \frac{\partial x}{b};\ \text{et}$$

$$\operatorname{Sec}\varphi^2 = \frac{a^2+b^2-2ax+x^2}{b^2}\ \text{erit igitur}$$

$$\text{I.}\ \frac{\partial\varphi \operatorname{Cot}\varphi^m}{(a+b\operatorname{Tang}\varphi)^n} = b^{m+1}\cdot\frac{\partial x}{x^n(x-a)^m(a^2+b^2-2ax+x^2)}.$$

$$\text{II.}\ \frac{\partial\varphi \operatorname{Tang}\varphi^m}{(a+b\operatorname{Tang}\varphi)^n} = \frac{1}{b^{m-1}}\cdot\frac{(x-a)^m\,\partial x}{x^n(a^2+b^2-2ax+x^2)}.$$

Omnes hujuscemodi exponentes differentiales erunt igitur pro quibuslibet numeris integris m, n per praecepta quarti capitis perfecte integrabiles.

Scholion.

Methodus, quam pro integratione exponentium differentialium trigonometricorum praescripsi, in eo consistit, ut per congruas substitutiones datis exponentibus forma exponentium differentialium algebraicorum tribuatur: quare cum haec transformatio semper succedat, cumque integrationem exponentium differentialium algebraicorum in praecedentibus capitibus ita pertractaverim, ut ea in quibuslibet casibus commodissime possit perfici, non est, cur alia methodus integrandi exponentes differentiales trigonometricos desideretur. Subjungam modo problema generale, ut ex ejus resolutione eluceat, quinam exponentes differentiales trigonometrici hac methodo possint reddi integrabiles.

339. Problema.

Integrare exponentem differentialem $X\partial\varphi$, *in quo praeter exponentem differentialem* $\partial\varphi$ *arcus* φ *certae hujus arcus functiones trigonometicae, ut sunt* Sin φ, Cos φ, Tang φ, Cot φ, Sinv φ, Cosv φ, Sec φ, Cosec φ, *et fors variae harum functionum potentiae, quomodocunque inter se et cum quantitatibus constantibus connexae, contineantur.*

Solutio.

1. Loco $\partial\varphi$ substituatur in dato exponente differentiali $X\partial\varphi$ aliquis valorum in (119. §.) determinatorum, qui videlicet ad transformationis simplicitatem plurimum conferre visus fuerit: introducetur eo ipso loco $\partial\varphi$ aliquis exponentium differentialium $\partial \operatorname{Sin}\varphi$, $\partial \operatorname{Cos}\varphi$, $\partial \operatorname{Tang}\varphi$, etc.

2. Ea

2. Ea ipsa jam functio trigonometrica arcus φ, nimirum Sin φ, vel Cos φ, aut Tang φ, etc., cujus exponens differentialis loco $\partial\varphi$ introductus fuerit (1), consideretur instar quantitatis variabilis x, aut summa vel differentia ejusdem functionis et certae quantitatis constantis, e. gr. $a \pm b\,\text{Sin}\,\varphi$, si quam datus exponens differentialis $X\partial\varphi$ complectitur, ponatur variabili x aequalis, prout illud aut hoc aptius visum fuerit; deinde exprimantur per variabilem x omnes reliquae functiones trigonometricae arcus φ secundum principia in (5. §. 1. Schol.) adducta.

3. Hac ratione reducetur integratio dati exponentis differentialis trigonometrici $X\partial\varphi$ ad integrationem certi exponentis $Z\partial x$ formae algebraicae, quae secundum principia superius exposita perficiatur.

340. Corollarium.

Si datus exponens differentialis $X\partial\varphi$ praeter exponentem differentialem $\partial\varphi$ arcus φ complectatur functiones arcus $m\varphi$, puta Sin $m\varphi$, Cos $m\varphi$, Tang $m\varphi$, etc., cum sit exponens differentialis $\partial . m\varphi = m\partial\varphi$; substituatur $\frac{\partial . m\varphi}{m}$ loco $\partial\varphi$, deinde peragantur reliqua, ut supra.

Exempla.

$$\text{Sit } \partial y = \partial\varphi . \text{Sin}\, m\varphi . \text{Cos}\, m\varphi^n;$$

$$\text{erit } \partial y = \frac{1}{m}\partial m\varphi . \text{Sin}\, m\varphi . \text{Cos}\, m\varphi^n:$$

$$= -\frac{1}{m}\partial\,\text{Cos}\, m\varphi . \text{Cos}\, m\varphi^n \text{ (115. §.)}:$$

$$\text{igitur } y = C - \frac{\text{Cos}\, m\varphi^{n+1}}{m(n+1)}.$$

$$\text{Sit } \partial y = \frac{\partial\varphi}{\text{Cos}\, m\varphi^n};$$

$$\text{erit } \partial y = \frac{1}{m} \cdot \frac{\partial . m\varphi}{\text{Cos}\, m\varphi^n}; \text{ hinc per (119. §.)}$$

$$\partial y = \frac{1}{m} \cdot \frac{\partial\,\text{Sin}\, m\varphi}{\text{Cos}\, m\varphi^{n+1}} = \frac{\partial x}{\sqrt{(1-x^2)^{n+1}}} \text{ pro } x = \text{Sin}\, m\varphi.$$

341. Problema.

Integrare exponentes differentiales $\varphi^n\partial\varphi . \text{Sin}\,\varphi$, $\varphi^n\partial\varphi . \text{Cos}\,\varphi$.

Sol-

Solutio.

1. Ponatur $Z = \varphi^n$; erit $\partial Z = n\varphi^{n-1}\partial\varphi$, $\overset{2}{\partial}Z = n(n-1)\varphi^{n-2}.\partial\varphi^2$; $\overset{3}{\partial}Z = n(n-1)(n-2)\varphi^{n-3}.\partial\varphi^3$, etc.

2. Porro est $\int\partial\varphi.\mathrm{Sin}\,\varphi = -\int\partial\mathrm{Cos}\,\varphi$ (115. §.) $= -\mathrm{Cos}\,\varphi$; et $\int\partial\varphi.\mathrm{Cos}\,\varphi = \int\partial\,\mathrm{Sin}\,\varphi$ (115. §.) $= \mathrm{Sin}\,\varphi$.

3. Quodsi ergo ponatur $Y = \mathrm{Sin}\,\varphi$, aut $Y = \mathrm{Cos}\,\varphi$; obtinebuntur ex (317. §.) sequentia integralia:

$$\int\varphi^n\partial\varphi.\mathrm{Sin}\,\varphi = C - \varphi^n\mathrm{Cos}\,\varphi + n\varphi^{n-1}\mathrm{Sin}\,\varphi + n(n-1)\varphi^{n-2}\mathrm{Cos}\,\varphi$$
$$- n(n-1)(n-2)\varphi^{n-3}\mathrm{Sin}\,\varphi - n(n-1)(n-2)(n-3)\varphi^{n-4}\mathrm{Cos}\,\varphi$$
$$+ \text{- - - - - - - - - - - - - - - - -}$$

$$\int\varphi^n\partial\varphi.\mathrm{Cos}\,\varphi = C + \varphi^n\mathrm{Sin}\,\varphi + n\varphi^{n-1}\mathrm{Cos}\,\varphi - n(n-1)\varphi^{n-2}\mathrm{Sin}\,\varphi$$
$$- n(n-1)(n-2)\varphi^{n-3}\mathrm{Cos}\,\varphi + \text{- - - - - - - -}$$

Patet, ita alternare signa $+-$, ut post quosvis binos terminos contiguos positivos sequantur bini contigui negativi, et vicissim. Porro clarum est, series has eo solo casu posse abrumpi, si n est numerus integer positivus.

342. Problema.

Datis integralibus $\int e^{r\varphi}\partial\varphi.\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^{m-2}$, $\int e^{r\varphi}\partial\varphi.\mathrm{Cos}\,\varphi^m\,\mathrm{Sin}\,\varphi^{n-2}$, *pro basi e logarithmorum naturalium, invenire integrale* $\int e^{r\varphi}\partial\varphi.\mathrm{Sin}\,\varphi^n\mathrm{Cos}\,\varphi^m$.

Solutio.

1. Ob $\partial.\frac{1}{r}e^{r\varphi} = e^{r\varphi}\partial\varphi$ (113. §.); debet esse integrale $\int e^{r\varphi}\partial\varphi = \frac{1}{r}e^{r\varphi}$ (239. §.).

2. Hinc (1) per (257. §.) elicies sequentem expressionem quaesiti integralis.

$$\int e^{r\varphi}\partial\varphi.\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m =$$
$$= \frac{1}{r}e^{r\varphi}\mathrm{Sin}\,\varphi^n\mathrm{Cos}\,\varphi^m - \frac{1}{r}\int e^{r\varphi}\partial\varphi\left(n\mathrm{Cos}\,\varphi^{m+1}\mathrm{Sin}\,\varphi^{n-1} - m\mathrm{Sin}\,\varphi^{n+1}\mathrm{Cos}\,\varphi^{m-1}\right).$$

3. Si porro partem alteram hujus aequationis resolvas per (257. §.), assumto integrali (1), tum posito $1 - \mathrm{Sin}\,\varphi^2 = \mathrm{Cos}\,\varphi^2$, et $1 - \mathrm{Cos}\,\varphi^2 = \mathrm{Sin}\,\varphi^2$, obtinebis

$$\int e^{r\varphi}\partial\varphi \,.\, \mathrm{Sin}\,\varphi^n \,\mathrm{Cos}\,\varphi^m = \frac{1}{r} e^{r\varphi}\,\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m$$

$$- \frac{1}{r^2} e^{r\varphi}\left(n\,\mathrm{Cos}\,\varphi^{m+1}\,\mathrm{Sin}\,\varphi^{n-1} - m\,\mathrm{Sin}\,\varphi^{n+1}\,\mathrm{Cos}\,\varphi^{m-1}\right)$$

$$+ \frac{1}{r^2}\int e^{r\varphi}\partial\varphi\left(m(m-1)\,\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^{m-2} + n(n-1)\,\mathrm{Cos}\,\varphi^m\,\mathrm{Sin}\,\varphi^{n-2}\right.$$

$$\left. - (n^2+m^2+2mn)\,\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m\right).$$

Quapropter debet esse

$$\int e^{r\varphi}\partial\varphi \,.\, \mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m =$$

$$= \frac{e^{r\varphi}(r\,\mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^m + m\,\mathrm{Sin}\,\varphi^{n+1}\,\mathrm{Cos}\,\varphi^{m-1} - n\,\mathrm{Cos}\,\varphi^{m+1}\,\mathrm{Sin}\,\varphi^{n-1})}{r^2+m^2+n^2+2mn}$$

$$+ \frac{m(m-1)}{r^2+m^2+n^2+2mn}\int e^{r\varphi}\partial\varphi \,.\, \mathrm{Sin}\,\varphi^n\,\mathrm{Cos}\,\varphi^{m-2}$$

$$+ \frac{n(n-1)}{r^2+m^2+n^2+2mn}\int e^{r\varphi}\partial\varphi\,\mathrm{Cos}\,\varphi^m\,\mathrm{Sin}\,\varphi^{n-2}$$

343. Corollarium 1.

Pro $n=0$, $m=0$, aut $n=0$, $m=1$, vel $m=0$, $n=1$, obtinebis ex (342. §.) sequentia integralia

$$\text{I.}\int e^{r\varphi}\partial\varphi = \frac{1}{r}e^{r\varphi} + C.$$

$$\text{II.}\int e^{r\varphi}\partial\varphi \,.\, \mathrm{Cos}\,\varphi = \frac{e^{r\varphi}(r\,\mathrm{Cos}\,\varphi + \mathrm{Sin}\,\varphi)}{r^2+1} + C.$$

$$\text{III.}\int e^{r\varphi}\partial\varphi \,.\, \mathrm{Sin}\,\varphi = \frac{e^{r\varphi}(r\,\mathrm{Sin}\,\varphi - \mathrm{Cos}\,\varphi)}{r^2+1} + C.$$

344. Corollarium 2.

Si ponas $m=0$, invenies per (342. §.) sequentem formulam, cujus ope poterit integrari exponens differentialis $e^{r\varphi}\partial\varphi \,.\, \mathrm{Sin}\,\varphi^k$ pro quolibet numero integro positivo k.

$$\int e^{r\varphi}\partial\varphi\,\mathrm{Sin}\,\varphi^n = \frac{e^{r\varphi}(r\,\mathrm{Sin}\,\varphi^n - n\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^{n-1})}{r^2+n^2}$$

$$= \frac{n(n-1)}{r^2+n^2}\int e^{r\varphi}\partial\varphi\,\mathrm{Sin}\,\varphi^{n-2}.$$

345. Corollarium 3.

Si vero fiat $n=0$ in (342. §.), obtinebitur sequens formula, ex qua integrale exponentis differentialis $e^{r\varphi}\partial\varphi.\mathrm{Cos}\,\varphi^m$ pro quolibet numero integro et positivo k poterit derivari.

$$\int e^{r\varphi}\partial\varphi\,\mathrm{Cos}\,\varphi^m = \frac{e^{r\varphi}(r\,\mathrm{Cos}\,\varphi^m + m\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^{m-1})}{r^2+m^2}$$
$$+\frac{m(m-1)}{r^2+m^2}\int e^{r\varphi}\partial\varphi\,\mathrm{Cos}\,\varphi^{m-2}.$$

346. Corollarium 4.

Pro $m=1$, vel $n=1$ in (342. §.) reperientur sequentes formulae, ope quarum integrari poterunt exponentes differentiales $e^{r\varphi}\partial\varphi\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^k$, $e^{r\varphi}\partial\varphi\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^k$ pro quolibet numero integro et positivo k.

$$\text{I.}\int e^{r\varphi}\partial\varphi.\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^n =$$
$$=\frac{e^{r\varphi}(r\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^n - n\,\mathrm{Cos}\,\varphi^2\,\mathrm{Sin}\,\varphi^{n-1} + \mathrm{Sin}\,\varphi^{n+1})}{r^2+n^2+2n+1}$$
$$+\frac{n(n-1)}{r^2+n^2+2n+1}\int e^{r\varphi}\partial\varphi.\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^{n-2}.$$

$$\text{II.}\int e^{r\varphi}\partial\varphi.\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^m =$$
$$=\frac{e^{r\varphi}(r\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^m + m\,\mathrm{Sin}\,\varphi^2\,\mathrm{Cos}\,\varphi^{m-1} - \mathrm{Cos}\,\varphi^{m+1})}{r^2+m^2+2m+1}$$
$$+\frac{m(m-1)}{r^2+m^2+2m+1}\int e^{r\varphi}\partial\varphi.\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^{m-2}$$

347. Corollarium 5.

Cum sit $\mathrm{Sin}\,\varphi^2 = 1-\mathrm{Cos}\,\varphi^2$, et $\mathrm{Cos}\,\varphi^2 = 1-\mathrm{Sin}\,\varphi^2$; poterit quodlibet productum $\mathrm{Sin}\,\varphi^u\,\mathrm{Cos}\,\varphi^v$ explicari per seriem formae $\mathrm{Cos}\,\varphi^v + A\,\mathrm{Cos}\,\varphi^{v+2} + B\,\mathrm{Cos}\,\varphi^{v+4} + \text{- - -} + P\,\mathrm{Cos}\,\varphi^{v+u}$, vel per seriem aliquam formae $\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^v + A\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^{v+2} + B\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^{v+4} + \text{- - - - - -} + P\,\mathrm{Sin}\,\varphi\,\mathrm{Cos}\,\varphi^{v+u-1}$, prout fuerit u numerus par vel impar: aut per seriem formae $\mathrm{Sin}\,\varphi^u + A\,\mathrm{Sin}\,\varphi^{u+2} + B\,\mathrm{Sin}\,\varphi^{u+4} + \text{- - -} + P\,\mathrm{Sin}\,\varphi^{u+v}$, vel $\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^u + A\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^{u+2} + \text{- - - - - - -} + P\,\mathrm{Cos}\,\varphi\,\mathrm{Sin}\,\varphi^{u+v-1}$. Licebit ergo exponentem differentialem $e^{r\varphi}\partial\varphi\,\mathrm{Sin}\,\varphi^u\,\mathrm{Cos}\,\varphi^v$ pro quibuslibet numeris integris et positivis u, v per (346. 345. 344. §.) perfecte integrare.

348. Problema.

Dato integrali $\int X\partial x$, *invenire integralia* $\int X\partial x \,\text{Arc Sin}\, x$, $\int X\partial x \,\text{Arc Cof}\, x$, $\int X\partial x \,\text{Arc Tang}\, x$.

Solutio.

Per (257. 120. 122. §.) invenies fequentes expreffiones pro integralibus, quae defiderantur.

$$\text{I. } \int X\partial x \,\text{Arc Sin}\, x = \text{Arc Sin}\, x \int X\partial x - \int \frac{\partial x \int X\partial x}{\sqrt{(1-x^2)}}.$$

$$\text{II. } \int X\partial x \,\text{Arc Cof}\, x = \text{Arc Cof}\, x \int X\partial x + \int \frac{\partial x \int X\partial x}{\sqrt{(1-x^2)}}.$$

$$\text{III. } \int X\partial x \,\text{Arc Tang}\, x = \text{Arc Tang}\, x \int X\partial x - \int \frac{\partial x \int X\partial x}{1+x^2}.$$

Quare, fi generatim integrale $\int X\partial x$ fuerit functio algebraica variabilis x, pendebunt haec integralia ab integratione exponentium differentialium algebraicorum, de qua fuperius tractavimus.

349. Problema.

Invenire functionem rationalem variabilis z *aequalem exponenti differentiali* $X\partial x$, *in quo functio irrationalis* $\sqrt{(a+bx\pm cx^2)}$ *cum variabili* x *ejusdemque functionibus rationalibus quoquo modo connexa continatur.*

Solutio.

Praeftabitur id, fi in dato exponente differentiali $X\partial x$ loco x, ∂x, $\sqrt{(a+bx\pm cx^2)}$ fequentes valores fubftituantur.

I. Pro pofitivo coefficiente quadrati x^2.

$$\sqrt{(a+bx+cx^2)} = \frac{bz+(a+z^2)\sqrt{c}}{b+2z\sqrt{c}};\quad x = \frac{z^2-a}{b+2z\sqrt{c}};$$

$$\partial x = \frac{2(bz+(a+z^2)\sqrt{c})\partial z}{(b+2z\sqrt{c})^2};\quad \text{pro } z = x\sqrt{c}+\sqrt{(a+bx+cx^2)}.$$

II. Pro negativo coefficiente quadrati x^2.

$$\sqrt{(a+bx-cx^2)} = \frac{z\sqrt{(b^2+4ac)}}{(1+z^2)\sqrt{c}};\quad x = \frac{b(1+z^2)+(1-z^2)\sqrt{(b^2+4ac)}}{2c(1+z^2)};$$

$$\partial x = \frac{-2z\partial z\sqrt{(b^2+4ac)}}{c(1+z^2)^2};\quad \text{pro } z = \frac{2\sqrt{c}\sqrt{(a+bx-cx^2)}}{\sqrt{(b^2+4ac)}-b+2cx}.$$

350. Co-

350. Corollarium 1.

Si datus exponens differentialis $X\partial x$ solam functionem irrationalem $\sqrt{(a \pm cx^2)}$, quomodocunque cum variabili x et ejus functionibus rationalibus connexam, complectatur; valores, pro quibus, loco x, ∂x, $\sqrt{(a \pm cx^2)}$ substitutis, ille formam rationalem induet, erunt, ob (349. §.) pro $b = 0$, sequentes.

I. Pro coefficiente positivo quadrati x^2.

$$\sqrt{(a+cx^2)} = \frac{z^2+a}{2z}; \quad x = \frac{z^2-a}{2z\sqrt{c}};$$

$$\partial x = \frac{(z^2+a)\partial z}{2z^2\sqrt{c}} \text{ pro } z = x\sqrt{c} + \sqrt{(a+cx^2)}.$$

II. Pro coefficiente negativo quadrati x^2.

$$\sqrt{(a-cx^2)} = \frac{2z\sqrt{a}}{1+z^2}; \quad x = \frac{(1-z^2)\sqrt{a}}{(1+z^2)\sqrt{c}};$$

$$\partial x = \frac{-4z\partial z\sqrt{a}}{(1+z^2)^2\sqrt{c}}; \text{ pro } z = \frac{\sqrt{(a-cx^2)}}{\sqrt{a}+x\sqrt{c}}.$$

351. Corollarium 2.

Per (350. §.) determinari possunt sequentia memorabilia integralia.

Posito $z = x + \sqrt{(\alpha + x^2)}$.

$$\int (x+\sqrt{(\alpha+x^2)})^n \partial x =$$

$$= \tfrac{1}{2}\int z^n \partial z + \frac{\alpha}{2}\int z^{n-2}\partial z = \frac{z^{n+1}}{2(n+1)} + \frac{\alpha z^{n-1}}{2(n-1)} =$$

$$= \frac{(x+\sqrt{(\alpha+x^2)})^{n+1}}{2(n+1)} + \frac{\alpha(x+\sqrt{(\alpha+x^2)})^{n-1}}{2(n-1)} + C.$$

Unde pro $n = -1$ obtinetur per (246. §.)

$$\int \frac{\partial x}{x+\sqrt{(\alpha+x^2)}} = \tfrac{1}{2}l(x+\sqrt{(\alpha+x^2)}) - \frac{\alpha}{4(x+\sqrt{(\alpha+x^2)})^2} + C.$$

$$\int (ex + c\sqrt{(\alpha+x^2)})(x+\sqrt{(\alpha+x^2)})^n \partial x =$$

$$= \tfrac{1}{4}(e+c)\int z^{n+1}\partial z + \frac{\alpha c}{2}\int z^{n-1}\partial z + \frac{\alpha^2}{4}(c-e)\int z^{n-3}\partial z =$$

$$= \frac{(e+c)(x+\sqrt{(\alpha+x^2)})^{n+2}}{4(n+2)} + \frac{\alpha c(x+\sqrt{(\alpha+x^2)})^n}{2n} + \frac{\alpha^2(c-e)(x+\sqrt{(\alpha+x^2)})^{n-2}}{4(n-2)} + C.$$

 Hinc

355. Problema.

Invenire functionem rationalem variabilis z aequalem exponenti differentiali $X \frac{\partial x}{x}$ complectenti functionem irrationalem $\sqrt[\mu]{\frac{a+bx^n}{f+gx^n}}$ cum rationalibus functionibus potentiae x^n quacunque ratione connexam.

Solutio.

Substituantur in dato exponente differentiali $X \frac{\partial x}{x}$ loco, x^n, $\frac{\partial x}{x}$, et $\sqrt[\mu]{\frac{a+bx^n}{f+gx^n}}$ sequentes valores.

$$x^n = \frac{fz^\mu - a}{b - gz^\mu}; \quad \frac{\partial x}{x} = \frac{\mu(bf - ag)z^{\mu-1}.\partial z}{n(fz^\mu - a)(b - gz^\mu)};$$
$$\text{pro } z = \sqrt[\mu]{\frac{a+bx^n}{f+gx^n}}.$$

356. Corollarium 1.

Pro $n=1$ obtinebuntur ex (355. §.) sequentes valores, qui, loco x, ∂x, et $\sqrt[\mu]{\frac{a+bx}{f+gx}}$, assumti, quemlibet exponentem differentialem $X \partial x$ reddent rationalem, si is solam functionem irrationalem $\sqrt[\mu]{\frac{a+bx}{f+gx}}$, quacunque ratione connexam cum rationalibus functionibus variabilis x, complectatur.

$$x = \frac{fz^\mu - a}{b - gz^\mu}; \quad \partial x = \frac{\mu(bf - ag)z^{\mu-1}.\partial z}{(b - gz^\mu)^2};$$
$$\text{pro } z = \sqrt[\mu]{\frac{a+bx}{f+gx}}.$$

357. Corollarium 2.

Si autem fiat $\mu = 2$, prodibunt ex (355. §.) sequentes expressiones, quae valores in exponente differentiali $X \frac{\partial x}{x}$ substituendos exhibent, ut is eo casu, quo X binas functiones irrationales $\sqrt{(a+bx^n)}$, $\sqrt{(f+gx^n)}$, solis constantibus a, b, f, g a se invicem discrepantes, et cum functionibus rationalibus potentiae x^n quoquo modo connexas complectitur, in rationalem possit converti.

x^n

$$x^n = \frac{fz^2 - a}{b - gz^2}; \quad \frac{\partial x}{x} = \frac{2(bf - ag)z\partial z}{n(fz^2 - a)(b - gz^2)};$$

$$\sqrt{(a + bx^n)} = \frac{z\sqrt{(bf - ag)}}{\sqrt{(b - gz^2)}}; \quad \sqrt{(f + gx^n)} = \frac{\sqrt{(bf - ag)}}{\sqrt{(b - gz^2)}};$$

$$\text{pro } z = \frac{\sqrt{(a + bx^n)}}{\sqrt{(f + gx^n)}} = \frac{\sqrt{(a + bx^n)(f + gx^n)}}{f + gx^n}.$$

Hac enim substitutione fiet, ut novus exponens differentialis unicam functionem irrationalem $\sqrt{(b - gz^2)}$ complectatur, de cujus eliminatione superius est actum.

358. Corollarium 3.

Et pro $n = 1$ in (357. §.) reperientur sequentes valores, qui in dato exponente irrationali $X\partial x$ substituti ipsum reddent rationalem, quoties ille binas functiones irrationales $\sqrt{(a + bx)}$ et $\sqrt{(f + gx)}$, cum variabili x ejusdemque functionibus rationalibus quocunque modo connexas, continuerit.

$$x = \frac{fz^2 - a}{b - gz^2}; \quad \partial x = \frac{2(bf - ag)z\partial z}{(b - gz^2)^2};$$

$$\sqrt{(a + bx)} = \frac{z\sqrt{(bf - ag)}}{\sqrt{(b - gz^2)}}; \quad \sqrt{(f + gx)} = \frac{\sqrt{(bf - ag)}}{\sqrt{(b - gz^2)}};$$

$$\text{pro } z = \frac{\sqrt{(a + bx)}}{\sqrt{(f + gx)}} = \frac{\sqrt{(a + bx)(f + gx)}}{f + gx}.$$

Novus enim exponens differentialis continebit unicam functionem irrationalem $\sqrt{(b - gz^2)}$, de cujus eliminatione superius est actum.

359. Problema.

Datum exponentem differentialem $X\partial x$, si is completam integrationem non admittit, integrare per seriem infinitam.

Solutio.

Explicetur functio X, secundum principia primi capitis, per seriem formae $Ax^a + Bx^b + Cx^c +$ etc., ut fiat $X\partial x = (Ax^a + Bx^b + Cx^c + \text{etc.})\partial x$; deinde fiat integratio per (249. §.): obtinebitur pro quaesito integrali $\int X\partial x$ series in infinitum excurrens, quae idcirco, quia tota haberi non potest, perfectum integrale haud dabit, quotquot ejus termini sumantur. Clarum tamen est, adproximari posse hac via ad verum integrale, quod in adplicatione calculi integralis aeque est utile, et necessarium; ac

adproximare ad radices numerorum, quorum perfectae radices nequeunt assignari.

360. Corollarium 1.

Si functio $X = \sqrt{(ax^m + bx^n)} = (ax^m + bx^n)^{\frac{1}{2}}$ juxta (54 §.) explicetur per seriem, obtinebitur per (249. §.) sequens notatu dignum integrale.

$$\int \partial x \sqrt{(ax^m + bx^n)} =$$

$$= 2\left(\frac{a^{\frac{1}{2}}.x^{\frac{m+2}{2}}}{m+2} + \frac{b.x^{\frac{2n-m+2}{2}}}{2a^{\frac{1}{2}}(2n-m+2)} - \frac{1.b^2.x^{\frac{4n-3m+2}{2}}}{2.4a^{\frac{3}{2}}(4n-3m+2)} \right.$$

$$+ \frac{1.3b^3.x^{\frac{6n-5m+2}{2}}}{2.4.6a^{\frac{5}{2}}(6n-5m+2)} - \frac{1.3.5b^4.x^{\frac{8n-7m+2}{2}}}{2.4.6.8a^{\frac{7}{2}}(8n-7m+2)}$$

$$+ - - - - - - - - - - - - - - - - - -$$

$$\left. \pm \frac{1.3.5.7 - - - - (2r-3)b^r.x^{\frac{2rn-(2r-1)m+2}{2}}}{2.4.6.8 - - - - 2r.a^{\frac{2r-1}{2}}(2rn-(2r-1)m+2)} \right) + C.$$

Terminus ultimus, utpote *rtus*, dabit singulos, post primum ordine sequentes, terminos, si loco r termini seriei 1, 2, 3, 4, 5, etc. successive substituantur.

361. Corollarium 2.

Et si $X = (ax^m + bx^n)^{-\frac{1}{2}}$ explicetur per seriem (55. §.); obtinebitur per (249. §.) sequens integrale.

$$\int \frac{\partial x}{\sqrt{(ax^m + bx^n)}} =$$

$$= 2\left(\frac{-1}{(m-2)a^{\frac{1}{2}}.x^{\frac{m-2}{2}}} + \frac{1.b}{2(3m-2n-2)a^{\frac{3}{2}}.x^{\frac{3m-2n-2}{2}}} \right.$$

$$- \frac{1.3b^2}{2.4(5m-4n-2)a^{\frac{5}{2}}.x^{\frac{5m-4n-2}{2}}} + \frac{1.3.5b^3}{2.4.6(7m-6n-2)a^{\frac{7}{2}}.x^{\frac{7m-6n-2}{2}}}$$

$$- - - - - - - - - + - - - - - - - - - - - -$$

$$\left. \pm \frac{1.3.5.7 - - - - - - - - - - - - - - (2r-1)b^r}{2.4.6.8 - - - 2r((2r+1)m-2rn-2)a^{\frac{2r+1}{2}}.x^{\frac{(2r+1)m-2rn-2}{2}}} \right) + C.$$

Scholion 1.

Nonnunquam commodius perficitur adproximatio, si, assumtis indeterminatis coefficientibus A, B, C, D, etc. ipsum integrale quaesitum $\int X \partial x$ ponatur aequari seriei $Ax + Bx^2 + Cx^3 + Dx^4 +$ etc., aut alteri formae similis, deinde ex aequatione per differentiationem producta, $X \partial x = \partial x (A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.})$ valores pro A, B, C, D, etc. secundum principia primi capitis determinentur. Saepenumero etiam, inventa aequatione $X = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.} = S$, logarithmi $\log X = \log S$ utilissime capientur, unde ulteriori differentiatione aequatio $\frac{\partial X}{X} = \frac{\partial S}{S}$ derivabitur, quae quaesitos valores A, B, C, D, etc. debebit prodere.

Haec methodus quaerendi aequationem pro determinatione valorum, quos coefficientes indeterminati debent obtinere, tunc praecipue notabili compendio calculi potest adhiberi, si functio X in dato exponente differentiali $X \partial x$ potentias polynomiorum complectatur.

E. gr. Si u, v sint certae functiones variabilis x, petaturque integrale exponentis differentialis $\frac{u^m}{v^n} \partial x$, quod seriei $Ax + Bx^2 + Cx^3 + Dx^4 +$ etc. aequari debeat: cum eo ipso fiat $\frac{u^m}{v^n} = A + 2Bx + 3Cx^2 + 4Dx^3 +$ etc., posita hac serie $= S$, erit

$$\frac{u^m}{v^n} = S, \text{ hinc } \log \frac{u^m}{v^n} = \log S; \text{ seu}$$

$$m\,l\,u - n\,l\,v = lS: \text{ igitur debet esse}$$

$$\frac{m \partial u}{u} - \frac{n \partial v}{v} = \frac{\partial S}{S},$$

unde valores coefficientium A, B, C, D, etc. quaeri poterunt.

Scholion 2.

Haec vero per coefficientes indeterminatos integrandi methodus (1. Schol.) id habet incommodi, quod, nisi exponentes variabilis x certa lege crescant, aut decrescant, inaniter permulti coefficientes introducan-[illegible] nas $\int \frac{\partial x}{1 - x^5 + x^7} = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5$ $A = 1$, $B = 0$, $C = 0$, $D = 0$, $E = 0$. Quocirca,

 ut

ut id videtur, confusum erit, quia non facile perspicitur, qua lege exponentes sese excipere debeant, eos interea indeterminatos relinquere, atque ex finali aequatione, unde jam valores ipsorum coefficientium eliciendi fuerint, ita determinare, ut ad illos principia in (28. 29. §.) stabilita possint adplicari. Ceterum, prouti in (516. §. 2. Schol.) tacite monui, nunquam hac aut alia methodo integrandi per series in indefinitum excurrentes utemur, si alia via superest, qua completum integrale dati exponentis differentialis per quamcunque demum functionem, algebraicam, logarithmicam, trigonometricam, aut mixtam, invenire licet: ubi vero ad ejusmodi methodum refugiendum fuerit, curabimus, ut series, terminis sequentibus continuo decrescentibus, convergant, divergentesque series, in quibus sequentes termini fiunt majores praecedentibus, omni casu vitabimus. Quo enim magis convergit series, quaesito integrali aequata, eo pauciores termini primi sufficiunt ad integrale prope verum exhbendum. Ex hac potissimum ratione seriebus formae $Ax^a + Bx^b + Cx^c +$ etc. aliae series formae generalis $Ax^a(\alpha + \beta x^n)^p + Bx^b(\alpha + \beta x^n)^q +$ etc. in multis disquisitionibus possunt substitui, cujus integrandi methodi exempla in sequentibus occurrent.

CAPUT VII.

DE INTEGRATIONE EXPONENTIUM AEQUATIONUMQUE DIFFERENTIALIUM PRIMI ALTIORISQUE ORDINIS PLURES VARIABILES COMPLECTENTIUM.

362. Definitio.

Integratio exponentis differentialis εy involventis duas aut plures variabiles u, v, z, earumque exponentes differentiales εu, εv, εz, absolvitur inventione functionis $Z+C$ variabilium u, v, z, cujus ratio differentialis determinata per (142. §.) habet exponentem dato εy aequalem, atque haec functio $Z+C$, in qua C denotat constantem indeterminatam, vocatur *completum integrale* exponentis differentialis εy. Integrale ejusmodi licebit aut simpliciter signo $\int \varepsilon y$, vel signo ${}^{uvz}\!\int \varepsilon y$ exprimere: quocirca poterit signum ${}^{r}\!\int Y \varepsilon r$, et quodvis ei simile, indicare integrale exponentis differentialis $V \varepsilon r$ spectati instar unius functionis solius variabilis r, licet fors V alias praeterea variabiles et earum exponentes differentiales complectatur.

Sic e. gr. pro $\varepsilon u = z x^2 \varepsilon x$, et $\varepsilon v = z^2 \varepsilon z \varepsilon x$ erit

$${}^{x}\!\int \varepsilon u = {}^{x}\!\int z x^2 \varepsilon x = \tfrac{1}{3} z x^3, \text{ et } {}^{z}\!\int \varepsilon v = {}^{z}\!\int z^2 \varepsilon z \varepsilon x = \tfrac{1}{3} z^3 \varepsilon x.$$

363. Corollarium 1.

Nullus exponens differentialis ${}^{uv}\varepsilon y$ involvens duas variabiles u, v erit per se integrabilis, nisi is aequetur certae functioni variabilium u, v formae $M \varepsilon u + N \varepsilon v$, in qua fit ${}^{v}\varepsilon M \varepsilon u = {}^{u}\varepsilon N \varepsilon v$ (145. 362. §.).

364. Corollarium 2.

Dato autem exponente differentiali $\varepsilon y = M \varepsilon u + N \varepsilon v$, in quo fit ${}^{v}\varepsilon M \varepsilon u = {}^{u}\varepsilon N \varepsilon v$, captoque integrali ${}^{u}\!\int M \varepsilon u$ (362. §.), quod differentiatum pro utraque variabili u, v per (142. §.) det exponentem ${}^{uv}\varepsilon . {}^{u}\!\int M \varepsilon u = M \varepsilon u + n \varepsilon v$, ita ut fit ${}^{v}\varepsilon M \varepsilon u = {}^{u}\varepsilon n \varepsilon v$ (145. §.); erit ${}^{u}\varepsilon (N - n) = 0$,

 adeo-

adeoque $N-n$ aut quantitas constans, aut aliqua functio solius variabilis v, carens variabili u: omnis ergo exponens differentialis $sy = Msu + Nsv$, in quo fuerit ${}^{v}sMsu = {}^{u}sNsv$, habebit pro integrali completo (362. §.) sequentem functionem.

$$y = {}^{u}\!\!\int Msu + {}^{v}\!\!\int (sy - {}^{uv}s \,.\, {}^{u}\!\!\int Msu) + C =$$
$$= {}^{u}\!\!\int Msu + {}^{v}\!\!\int \left(N - \frac{{}^{v}s \,.\, {}^{u}\!\!\int Msu}{sv}\right) sv + C.$$

365. Corollarium 3.

Si detur exponens differentialis ${}^{uvz}sy$ involvens tres variabiles u, v, z, non erit is per se integrabilis, nisi sub forma $Usu + Vsv + Zsz$, in qua sit ${}^{v}sUsu = {}^{u}sVsv$, ${}^{z}sUsu = {}^{u}sZsz$, ${}^{z}sVsv = {}^{v}sZsz$ (149. 362. §.).

366. Corollarium 4.

Quodsi vero in exponente differentiali trium variabilium formae $sy = Usu + Vsv + Zsz$ sit ${}^{v}sUsu = {}^{u}sVsv$, ${}^{z}sUsu = {}^{u}sZsz$, ${}^{z}sVsv = {}^{v}sZsz$, capto integrali ${}^{u}\!\!\int Usu$ (362. §.) quod differentiatum pro omnibus variabilibus u, v, z, dabit certum exponentem differentialem ${}^{uvz}s \,.\, {}^{u}\!\!\int Usu = Usu + Psv + Qsz$, ita ut sit ${}^{v}sUsu = {}^{u}sPsv$, ${}^{z}sUsu = {}^{u}sQsz$, ${}^{z}sPsv = {}^{v}sQsz$ (149. §.); erit eo ipso $sy - {}^{uvz}s \,.\, {}^{u}\!\!\int Usu = (V-P)sv + (Z-Q)sz$, et ${}^{z}s(V-P)sv = {}^{v}s(Z-Q)sz$, existente ${}^{u}s(V-P) = o$, et ${}^{u}s(Z-Q) = o$, adeoque tam functione $V-P$ quam $Z-Q$ carente variabili u: integrale completum ejusmodi exponentis differentialis sy comprehendetur ergo sequenti formula (362. §.), cujus pars prima per praecepta praecedentium capitum, pars altera vero per (364. §.) est perfecte determinabilis.

$$y = {}^{u}\!\!\int Usu + {}^{vz}\!\!\int (sy - {}^{uvz}s \,.\, {}^{u}\!\!\int Usu) + C.$$

367. Corollarium 5.

Semper ergo, dum exponentes differentiales, involventes duas tresve variabiles earundemque exponentes differentiales, fuerint integrandi (362. §.), dispiciatur ante omnia, utrum ii sint per se integrabiles (363. 365. §.); tum, si sint per se integrabiles, quaerantur illorum integralia juxta (364. 366. §.): sic enim, ut per se patet, integratio illorum reducitur ad integrationem exponentium differentialium unicam variabilem involventium, de qua in praecedentibus capitibus est actum.

368. De-

368. Definitio.

Aequatio differentialis primi ordinis, vel simpliciter *Aequatio differentialis* vocabitur in sequentibus omnis aequatio inter duas aut plures variabiles u, v, z, earumque exponentes differentiales εu, εv, εz: aequatio autem inter variabiles u, v, z, earum exponentibus differentialibus harum variabilium, vocatur *Integrale datae aequationis differentialis*, si illa per (141. §.) differentiata restituat datam aequationem differentialem. Omnis porro aequatio differentialis formae $M\varepsilon u + N\varepsilon v + O\varepsilon z = 0$ erit *homogenea*, vel *heterogenea*: primum, si M, N, O sint totidem homogeneae ejusdemque dimensionis functiones variabilium u, v, z (12. 8. §.); alterum, si alterutrum aut utrumque horum desit. Dicentur demum variabiles u, v, z in aequatione differentiali $M\varepsilon u + N\varepsilon v + O\varepsilon z = 0$ *separatae* esse aut *permixtae*: primum, si M complectatur solam variabilem u, adeoque careat variabilibus v, z, N solam variabilem v, et O solam z; alterum vero, si aliquis factorum M, N, O involvat variabilem ab illa, in cujus exponentem differentialem is ductus est, distinctam.

369. Corollarium 1.

Omnis aequatio differentialis potest induere formam aequationis $\psi = 0$: completum vero ejus integrale, denotante C constantem indeterminatam, erit $Z = C$, si exponens differentialis functionis Z per (141. §.) pro omnibus variabilibus, quae in $\psi = 0$ continentur, determinatus exaequet functionem differentialem ψ (368. §.).

370. Corollarium 2.

Dato integrali $Z = \int \varepsilon y$ exponentis differentialis εy duas tresve variabiles involventis, sumtaque constante indeterminata C, erit $Z = C$ integrale completum aequationis differentialis $\varepsilon y = 0$ (369. 362. §.).

371. Corollarium 3.

Criterium generale integrabilitatis omnium aequationum differentialium $M\varepsilon u + N\varepsilon v = 0$ inter duas variabiles u, v in eo consistet, ut sit ${}^{v}\varepsilon M\varepsilon u = {}^{u}\varepsilon N\varepsilon v$: nulla enim aequatio differentialis hujus generis erit per se integrabilis, quae hac proprietate destituitur (368. 145. §.); quaecunque vero aequatio differentialis habuerit hanc proprietatem, erit illa eo ipso per se integrabilis, actuque integrabitur, si quaeratur integrale $Z = \int(M\varepsilon u$ $N\varepsilon v)$ per (364. §.), tum fiat $Z = C$ (370. §.).

372. Corollarium 4.

Et criterium integrabilitatis aequationum differentialium $U\,su + V\,sv + Z\,sz = 0$ inter tres variabiles u, v, z consistet in relationibus $^{v}s\,U\,su = {}^{u}s\,V\,sv$, $^{z}s\,U\,su = {}^{u}s\,Z\,sz$, $^{z}s\,V\,sv = {}^{v}s\,Z\,sz$: aequatio nimirum differentialis non erit per se integrabilis, nisi singulae hae relationes in ea locum habeant (368. 149. §.): quoties vero proposita fuerit aequatio differentialis singulas has relationes offerens; toties erit ea per se integrabilis, eritque ejus integrale completum $Z = C$, si $Z = \int(U\,su + V\,sv + Z\,sz)$ per (366. §.) determinetur (370. §.).

373. Corollarium 5.

Omnis aequatio differentialis $M\,su + N\,sv = 0$, vel $M\,su + N\,sv + O\,sz = 0$, in qua variabiles sunt separatae (368. §.), debet esse per se integrabilis, ita ut integrale completum prioris sit $^{u}\!\int M\,su + {}^{v}\!\int N\,sv = C$, et $^{u}\!\int M\,su + {}^{v}\!\int N\,sv + {}^{z}\!\int O\,sz = C$ posterioris (371. 372. §.).

374. Corollarium 6.

Si aequatio $M\,su + N\,sv = 0$, quae dat $\frac{su}{sv} = \frac{-N}{M}$, non sit arbitraria, sed in resolutione cujuspiam problematis legitimo ratiocinio determinata; necesse est, ut ea certae functioni $y = C$ variabilium u, v respondeat, pro qua debet necessario extare aequatio aliqua differentialis $P\,su + Q\,sv = {}^{uv}sy = 0$, ita ut sit $^{v}s\,P\,su = {}^{u}s\,Q\,sv$ (145. §.): igitur erit etiam $\frac{su}{sv} = \frac{-Q}{P}$, hinc $\frac{N}{M} = \frac{Q}{P}$; proinde vel est $N = Q$ et $M = P$, hinc quoque $^{v}s\,M\,su = {}^{u}s\,N\,sv$, vel debet dari tertia quaepiam functio F variabilium u, v, pro qua fieret $NF = Q$ et $MF = P$, hinc $^{v}s.\,MF\,su = {}^{u}s.\,NF\,sv$.

375. Corollarium 7.

Quamobrem omnis aequatio differentialis $M\,su + N\,sv = 0$, ad quam in resolutione alicujus problematis legitimo ratiocinio fuerit perventum, debebit esse per se integrabilis, aut, si non sit per se integrabilis, possibilis erit factor F, ex quo et data aequatione nasceretur aequatio identica $MF\,su + NF\,sv = 0$ per se integrabilis (371. 374. §.).

376. Corollarium 8.

Quaevis aequatio homogenea (368. §.) $M\partial u + N\partial v = 0$ variabiles u, v permixtas involvens est talis indolis, ut illa simplici substitutione novae variabilis $z = \frac{v}{u}$ in aequationem identicam variabiles u, z separatas involventem, adeoque per (373. §.) integrabilem possit transformari. Cum enim in hac hypothesi M, N sint functiones variabilium u, v et homogeneae et ejusdem puta *mtae* dimensionis (368. §.); necesse utique est, ut pro $v = uz$, hinc $\partial v = u\partial z + z\partial u$, M, N abeant in producta Pu^m, Qu^m ex certis functionibus P, Q solius variabilis z in potentiam u^m, dataque aequatio differentialis abeat in $Pu^m \partial u + Qu^m(u\partial z + z\partial u) = 0$, ex qua nascitur sequens aequatio differentialis variabiles u, z separatas complectens (368. §.), ejusque integralis per (373. §.).

$$\frac{\partial u}{u} + \frac{Q\partial z}{P+Qz} = 0; \quad lu + \int \frac{Q\partial z}{P+Qz} = C.$$

377. Corollarium 9.

Hoc modo poterit quoque aequatio homogenea $M\partial u + N\partial v + O\partial z = 0$ involvens tres variabiles permixtas u, v, z (368. §.) transformari in aliam, quae unam variabilem a reliquis duabus separatam complectatur, quo ipso ejus integratio ad integrationem exponentis differentialis involventis duas variabiles (362. §.) reducitur. Si enim M, N, O sint functiones homogeneae *mti* ordinis (368. 12. 8. §.), et fiat $v = u\varphi$, $z = u\mu$, hinc $\partial v = u\partial\varphi + \varphi\partial u$, $\partial z = u\partial\mu + \mu\partial u$; necesse est, ut pro his valoribus functiones M, N, O abeant in producta Pu^m, Qu^m, Ru^m pro certis functionibus P, Q, R, variabilium φ, μ carentibus tertia variabili u; data autem aequatio convertetur in $Pu^m\partial u + Qu^m(u\partial\varphi + \varphi\partial u) + Ru^m(u\partial\mu + \mu\partial u) = 0$, ex qua sequens prodit aequatio differentialis;

$$\frac{\partial u}{u} + \frac{Q\partial\varphi + R\partial\mu}{P + Q\partial\varphi + R\partial\mu} = 0.$$

378. Corollarium 10.

Licet ergo aliqua aequatio differentialis non sit per se integrabilis (371. 372. §.), inde nondum sequitur, eam absolute non esse integrabilem; fieri enim potest, ut certo modo transformata reddatur per se integrabilis, sicut e. gr. omnis aequatio homogenea in (376. §.).

379. Problema.

Integrare datam quamcunque aequationem differentialem $M \partial u + N \partial v = 0$.

Solutio.

1. Ante omnia dispiciatur, utrum ea sit per se integrabilis, et si est per se integrabilis, quaeratur ejus aequatio integralis per (371. §.).

2. Quodsi vero data aequatio differentialis non sit per se integrabilis (371. §.), videatur, an non certis artificiis ita possit transformari, ut eo ipso reddatur per se integrabilis (378. §.): generales regulae in eum finem nullae possunt praescribi; particulares autem methodi, quae in specialibus casibus succedunt, usu optime discuntur. Ceterum constat ex praecedentibus, omnem aequationem differentialem per se non integrabilem vel homogeneam esse vel heterogeneam; omnem porro aequationem homogeneam per (376. §.) posse reddi integrabilem, unde elucet, solas aequationes heterogeneas superesse, pro quibus desiderantur artificia, ope quorum eae reddi queant integrabiles.

3. Quare videatur, an non data aequatio heterogenea sic sit comparata, ut aut, retentis variabilibus u, v, vulgaribus operationibus algebraicis, aut substitutione novae cujuspiam variabilis sic possit transformari, ut transformata vel variabiles separatas complectatur, vel saltem sit homogenea: casu enim primo erit ea per (373. §.) integrabilis; et casu altero per (376. §.). Nonnunquam licebit factorem F quaerere, qui propositam aequationem haeterogeneam, si ea per illum multiplicetur, reddat per (371. §.) integrabilem (375. §.).

4. In desperatis casibus quaeratur series, quae variabilem u per alteram v ita exprimat, ut ea datae aequationi differentiali satisfaciat. Eapropter fingatur esse $u = A v^{\alpha} + B v^{\alpha+1} + C v^{\alpha+2} + \text{etc.}$, hinc $\partial u = (\alpha A v^{\alpha-1} + (\alpha+1) B v^{\alpha} + (\alpha+2) C v^{\alpha+1} + \text{etc.}) \partial v$, tum substituantur hi valores loco u et ∂u in data aequatione $M \partial u + N \partial v = 0$: hac enim ratione obtinebitur aequatio, ex qua, valore exponentis α rite determinato, valores coefficientium indeterminatorum A, B, C, etc. per (28. vel 29. §.) poterunt derivari.

380. Corollarium 1.

Exemplum aequationum heterogenearum per se non integrabilium (371. §.), quae simplici substitutione novae variabilis in aequationes identicas imme-

immediate per (373. §.) integrabiles possunt transformari (379. §. 3. n.), offert aequatio $A\partial u + m(Bu + Cv)\partial u - D\partial v + n(Bu + Cv)\partial v = o$: haec enim, pro $Bu + Cv = z$, hinc $\partial v = \frac{\partial z - B\partial u}{C}$, abit in sequentem aequationem differentialem, cujus integrale completum, ut sequitur, per (373. §.) licuit determinare.

$$\partial u + \frac{(nz - D)\partial z}{AC + BD + (mC - nB)z} = o.$$

$$\text{Const} = u + \frac{n(AC + BD + (mC - nB).z)}{(mC - nB)^2}$$

$$- \frac{D(mC - nB) + n(AC + BD)}{(mC - nB)^2} \log(AC + BD + (mC - nB)z).$$

381. Corollarium 2.

Unicus casus in (380. §.) debet excipi, nimirum quo fit $mC = nB$, adeoque $m = \frac{nB}{C}$: hoc autem casu determinato, ob $(mC - nB)z = o$, habebimus in (380. §.) $\partial u + \frac{(nz - D)\partial z}{AC + BD} = o$, adeoque per (373. §.) erit

aequatio integralis

$$u + \frac{n(Bu + Cv)^2 - 2D(Bu + Cv)}{2(AC + BD)} = \text{Const.}$$

aequationis differentialis

$$A\partial u + \frac{nB}{C}(Bu + Cv)\partial u - D\partial v + n(Bu + Cv)\partial v = o.$$

382. Corollarium 3.

Exemplum autem aequationum heterogenearum per se non integrabilium (371. §.), quae simili substitutione in aequationes homogeneas per (376. §.) integrabiles possunt transformari (379. §. 3. n.) habemus in aequatione $(A + Bu + Cv)\partial u - (D + Eu + Fv)\partial v = o$: si enim ponatur $A + Bu + Cv = \varphi$ et $D + Eu + Fv = \mu$, hinc

$$\partial u = \frac{F\partial\varphi - C\partial\mu}{BF - CE}; \quad \partial v = \frac{B\partial\mu - E\partial\varphi}{BF - CE};$$

abibit data aequatio in homogeneam identicam $(F\varphi + E\mu)\partial\varphi - (B\mu + C\varphi)\partial\mu = o$ (368. §.), ex qua pro $z = \frac{\varphi}{\mu} = \frac{A + Bu + Cv}{D + Eu + Fv}$ per

 (376.

(376. §.) obtinetur sequens aequatio integralis, complectens integrale per (373. §.) determinabile.

$$\log \mu + \int \frac{(E+Fz)\partial z}{Fz^2+(E-C)z-B} = \text{Const.}$$

Notetur tamen casus, quo fit $BF = CE$, adeoque $BF - CE = 0$ in expressionibus pro ∂u, ∂v: hoc enim casu abit data aequatio in $A\partial u + (Bu+Cv)\partial u - D\partial v - \frac{F}{C}(Bu+Cv)\partial v = 0$, quae per (380. 381. §.) est integrabilis.

383. Corollarium 4.

Exemplum porro aequationum heterogenearum per se non integrabilium (371. §.), quarum conditiones a priori licet determinare, sub quibus illae in homogeneas per (376. §.) integrabiles possint transformari (379. §. 3. n.), exhibet aequatio $(u^a v^b + u^c v^d)\partial u + u^e v^f \partial v = 0$. Si enim in hac ponatur $v = z^n$, obtinebitur aequatio $(u^a z^{nb} + u^c z^{nd})\partial u + n u^e z^{n(f+1)-1}\partial z = 0$, quae eo duntaxat casu erit homogenea, si fuerit $a+nb = c+nd = e+n(f+1)-1$ (368. 12. 8. §.), adeoque $(d-f-1)(a-c) = (e-c-1)(d-b)$, et $n = \frac{a-c}{d-b}$, unde sequitur: omnem aequationem heterogeneam $(u^a v^b + u^c v^d)\partial u + u^e v^f \partial v = 0$, in qua est $(d-f-1)(a-c) = (e-c-1)(d-b)$, transformabilem esse in homogeneam per (376. §.) integrabilem, nimirum pro $v = z^{\frac{a-c}{d-b}}$, exceptis, ut per se patet, duobus casibus, quorum primo fit $\frac{a-c}{d-b} = \frac{0}{d-b}$ pro $a = c$, et altero $\frac{a-c}{d-b} = \frac{a-c}{0}$ pro $d = b$: his autem casibus abit data aequatio in sequentes aequationes per (373. §.) integrabiles.

$$u^{a-e}\partial u + \frac{v^f \partial v}{v^b+v^d} = 0; \quad (u^{a-e}+u^{c-e})\partial u + v^{f-b}\partial v = 0.$$

384. Corollarium 5.

Integratio datae aequationis differentialis $M\partial u + N\partial v = 0$ per se non integrabilis (371. §.) ope factoris (379. §. 3. n.) exigit, ut determinetur functio F, pro qua aequatio identica $MF\partial u + NF\partial v = 0$ sit per se integrabilis (375. §.): quare, cum in hac hypothesi debeat fieri $(M^v\partial F + F^v\partial M)\partial u = (N^u\partial F + F^u\partial N)\partial v$ (371. §.); habebimus sequentem aequationem,

tionem, ex qua factor F deberet elici, si is *a priori* esset determinabilis; verum non facile difficultates superaveris, quibus haec disquisitio est obnoxia.

$$F\left(\frac{{}^{v}\partial M}{\partial v}-\frac{{}^{u}\partial N}{\partial u}\right)+M\frac{{}^{v}\partial F}{\partial v}-N\frac{{}^{u}\partial F}{\partial u}=0.$$

385. Corollarium 6.

Interea notetur casus peculiaris, quo est $\left(\frac{{}^{v}\partial M}{\partial v}-\frac{{}^{u}\partial N}{\partial u}\right):N$ certa functio solius variabilis u, carens variabili v. Cum enim in proposita aequatione $M\partial u+N\partial v=0$ debeat esse $M=-\frac{N\partial v}{\partial u}$, si hic valor substituatur in (384. §.), obtinebitur $\frac{{}^{uv}\partial F}{F}=U\partial u$, posito $U=\left(\frac{{}^{v}\partial M}{\partial v}-\frac{{}^{u}\partial N}{\partial u}\right):N.$ Quamobrem, quoties caruerit U variabili v, quo casu integrale $\int U\partial u$ per ea, quae in praecedentibus sunt capitibus tradita, poterit determinari; toties erit $\log F=\int U\partial u$, adeoque, sumta basi e logarithmorum naturalium, $F=e^{\int U\partial u}$ factor, pro quo data aequatio differentialis $M\partial u+N\partial v=0$ per se non integrabilis transformabitur in aequationem identicam $Me^{\int U\partial u}.\partial u+Ne^{\int U\partial u}.\partial v=0$ per se integrabilem; ipsum autem integrale erit idcirco ob (364. §.)

$${}^{v}\!\!\int Ne^{\int U\partial u}.\partial v+{}^{u}\!\!\int\left(Me^{\int U\partial u}-\frac{{}^{u}\partial\,{}^{v}\!\!\int Ne^{\int U\partial u}.\partial v}{\partial u}\right)\partial u=C.$$

386. Corollarium 7.

Exemplo sit aequatio differentialis $Pv\partial u+Q\partial u+\partial v=0$, complectens functiones quascunque P, Q variabilis u carentes variabili v. Posito $M=Pv+Q$, et $N=1$, adeoque ${}^{v}\!\!\int Ne^{\int U\partial u}.\partial v={}^{v}\!\!\int e^{\int P\partial u}.\partial v=ve^{\int P\partial u}$, hinc ${}^{u}\partial.\,{}^{v}\!\!\int Ne^{\int U\partial u}.\partial v=vPe^{\int U\partial u}.\partial u$, obtinebitur per (385. §.) $ve^{\int P\partial u}+{}^{u}\!\!\int\partial u\,Qe^{\int P\partial u}=C$ pro completo integrali illius aequationis differentialis.

387. Corollarium 8.

Per (386. §.) possunt integrari omnes aequationes differentiales formula generali $Uv\partial u+Vv^{n}\partial u+T\partial v=0$ comprehensae, modo functiones

nes U, V, T careant variabili v. Cum enim sit $\frac{U}{T} v \varepsilon u + \frac{V}{T} v^n \varepsilon u + \varepsilon v = 0$, posito $z = \frac{1}{v^{n-1}}$, obtinebitur aequatio $\varepsilon z - (n-1) \frac{U}{T} z \varepsilon u - (n-1) \frac{V}{T} \varepsilon u = 0$ per (386. §.) integrabilis; nisi fors sit $n = 1$, quo casu data aequatio differentialis abit in aequationem $\frac{U+V}{T} \varepsilon u + \frac{\varepsilon v}{v} = 0$ per (373. §.) integrabilem.

388. Definitio.

Aequatio differentialis secundi ordinis, seu *Aequatio differentio-differentialis* erit in sequentibus omnis aequatio inter variabiles quascunque u, v, z earumque exponentes differentiales $\overset{2}{\varepsilon} u$, $\overset{2}{\varepsilon} v$, $\overset{2}{\varepsilon} z$ secundi ordinis.

389. Corollarium 1.

Sicut exponentes differentiales $\overset{2}{\varepsilon} u$, $\overset{2}{\varepsilon} v$, $\overset{2}{\varepsilon} z$ secundi ordinis obtinentur differentiatione exponentium differentialium εu, εv, εz primi ordinis; sic pro omni aequatione differentiali secundi ordinis relata ad variabiles u, v, z poterit cogitari aequatio differentialis primi ordinis relata ad easdem variabiles, e cujus differentiatione resultet aequatio differentialis secundi ordinis datae identica (388. §.).

390. Corollarium 2.

Omnes aequationes differentio-differentiales inter variabiles u, v in duas classes possunt distribui: ad primam classem pertinent aequationes, in quibus alterutra variabilium u vel v pro absoluta sumitur (4. §.), quo exponens differentialis $\varepsilon u = 1$ vel $\varepsilon v = 1$ fiat constans (81. §.) et differentio-differentialis aequalis nihilo, nimirum $\overset{2}{\varepsilon} u = 0$ vel $\overset{2}{\varepsilon} v = 0$: ad alteram vero classem possunt referri aequationes, in quibus neutra variabilium u, v sumitur pro absoluta, quae idcirco utrumque exponentem differentio-differentialem $\overset{2}{\varepsilon} u$, $\overset{2}{\varepsilon} v$ involvunt.

391. Corollarium 3.

Dum petitur aequatio inter variabiles u, v carens exponentibus differentialibus harum variabilium, cujus secundum differentiale idem sit cum data

data aequatione differentio - differentiali, duplici integratione est' opus: *primum integrale* datae aequationis differentio - differentialis est aequatio differentialis primi ordinis, cujus differentiatio restituit datam aequationem differentio - differentialem; *secundum, integrale* datae aequationis differentio - differentialis est integrale ejus primi integralis (389. 368. §.).

392. Corollarium 4.

Est autem integratio aequationum differentialium primi ordinis semper in potestate, ita ut illa omni casu saltem per series possit perfici (379. §.): secunda integratio cujuslibet aequationis differentio - differentialis poterit ergo pro absoluta haberi, si prima illius integratio fuerit in potestate (391. §.).

393. Corollarium 5.

Ceterum facile ex hactenus dictis colligitur, integrale primum aequationis differentio - differentialis pendentis a variabilibus u, v, in qua variabilis v pro absoluta sumitur, constantem indeterminatam $C \partial v$ debere involvere, ut id completum sit (391. §.).

Scholion.

Desideratur itaque methodus inveniendi prima integralis aequationum differentio - differentialium (392. §.): generalem, quae omni casu succedat, nullam habemus: quare, ne extra fines egrediamur, quos in concinnandis his institutionibus debebamus constituere, indicia immediatae integrabilitatis aequationum differentio - differentialium, methodumque integrandi aequationes per se integrabiles, tum nonnulla artificia breviter explanabimus, quibus aequationes differentio - differentiales per se haud integrabiles sic possunt in certis casibus transformari, ut illae eo ipso reddantur integrabiles. Insigne ejusmodi transformationum exemplum offert aequatio $x^2(a+bx^n)\partial^2 y+x(c+ex^n)\partial y \partial x+(f+gx^n)y\partial x^2=0$, cujus integratio complures celeberrimos Analystas hactenus occupavit, a nemine vero adeo simplici, generali, perfectaque methodo pertractata est, ut ejus meditationes illis possint aequiparari, quas *Cl. Pfaff* in suis *disquisitionibus analyticis* recentissime exposuit. Verum haec interea intacta cogimur relinquere, expedituri in sequentibus, si iis opus habuerimus.

394. Pro-

394. Problema.

Explorare, utrum aequatio differentio-differentialis formae $\overset{2}{s}Z = M\overset{2}{s}u + N\overset{2}{s}v + Psusv + Qsu^2 + Rsv^2 = 0$ *sit per se integrabilis.*

Solutio.

Dispiciatur, an sit $Qsu = {}^{u}sM$, $Rsv = {}^{v}sN$, et $Psusv = {}^{v}sMsu + {}^{u}sNsv$: nisi enim hae relationes inter functiones M, N, P, Q, R locum habeant, aequatio proposita non erit per se integrabilis: pro iisdem vero relationibus erit $sZ = Msu + Nsv = 0$ primum integrale ejusdem aequationis (391. 392. 148. §.), ex quo per praecedentia praecepta secundum integrale $Z = C$ debebit derivari.

395. Corollarium 1.

Cum sit ${}^{u}sRsv = {}^{u}s.{}^{v}sN$ in (394. §.) $= {}^{v}s.{}^{u}sN$ (146. §.); erit in (394. §.) $Psusv = {}^{v}sMsu + sv{}^{v}\!\int{}^{u}sRsv$: si ergo fuerit v variabilis absoluta, proinde $sv = 1$ constans, et $\overset{2}{s}v = 0$; debebit fieri ${}^{v}sPsusv = {}^{v}\overset{2}{s}Msu + {}^{u}sRsv^2$.

396. Corollarium 2.

Aequatio autem $\overset{2}{s}Z = 0$ in (394. §.) erit per se primo- et secundo-integrabilis, id est, tam prima quam secunda integratio ejus aequationis (391. §.) poterit citra omnem praeviam transformationem perfici, si fuerit $Qsu = {}^{u}sM$, $Rsv = {}^{v}sN$, et $Psusv = 2\,{}^{v}sMsu = {}^{v}sNsv$: erit autem ejus secundum integrale per (394. 364. §.)

$$Z = {}^{u}\!\!\int Msu + {}^{v}\!\!\int\left(N - \frac{{}^{v}s\,{}^{u}\!\!\int Msu}{sv}\right)sv + C.$$

397. Corollarium 3.

Quodsi vero v pro variabili absoluta sumatur, quo $sv = 1$ fiat constans, et $\overset{2}{s}v = 0$, habebimus aequationem differentio-differentialem generalissimae formae $\overset{2}{s}Z = M\overset{2}{s}u + Psusv + Qsu^2 + Rsv^2 = 0$: omnis aequatio hujus formae, si fuerit $Qsu = {}^{u}sM$, et ${}^{v}sPsusv = {}^{v}\overset{2}{s}Msu + {}^{u}sRsv^2$, erit per se integrabilis, primumque ejus integrale exhibebit sequens formula (394. 395. 393. §.), quae, ut expendenti patet, ita intelligenda est, ut in illa pars tertia in certis casibus possit aequari nihilo.

$$sZ = Msu + sv{}^{v}\!\!\int Rsv + {}^{u}\!\!\int(\overset{2}{s}Z - {}^{uv}s(Msu + sv{}^{v}\!\!\int Rsv)) + Csv = 0.$$

398. Corollarium 4.

Haec tamen indicia nequaquam demonstrant, aequationem $\overset{2}{\varepsilon}Z=0$ (397. §.) esse quoque secundo-integrabilem: potest enim ea ob illa indicia esse per se primo-integrabilis, quin ideo sit per se etiam secundo-integrabilis (394. 397. §.). Sic e. gr. aequatio $(u^3-v)\overset{2}{\varepsilon}u+(v-1)\varepsilon u\varepsilon v+3u^2\varepsilon u^2+u\varepsilon v^2=0$ dat $M=u^3-v$, $P=v-1$, $Q=3u^2$, $R=u$, adeoque $Q\varepsilon u={}^{u}\varepsilon M$, et ${}^{v}\varepsilon P\varepsilon u\varepsilon v={}^{v}\overset{2}{\varepsilon}M\varepsilon u+{}^{u}\varepsilon R\varepsilon v^2$: ob has relationes est illa per se primo-integrabilis, sed non ideo per se etiam secundo-integrabilis: primum enim ejus integrale est (397. §.) aequatio differentialis $(u^3-v)\varepsilon u+vu\varepsilon v+C\varepsilon v=0$, per se haud integrabilis (363. §.).

399. Corollarium 5.

Indiciis quidem, quorum ope ex relationibus inter factores M, P, Q, R colligi queat, utrum proposita aequatio differentio-differentialis formae $\overset{2}{\varepsilon}Z=M\overset{2}{\varepsilon}u+P\varepsilon u\varepsilon v+Q\varepsilon u^2+R\varepsilon v^2=0$ sit per se tam primo- quam secundo-integrabilis, carere possumus (392. §.): ea tamen ex (394. 397. §.) ultro se offerunt, nimirum $Q\varepsilon u={}^{u}\varepsilon M$, ${}^{v}\varepsilon P\varepsilon u\varepsilon v=2.{}^{v}\overset{2}{\varepsilon}M\varepsilon u=2.{}^{u}\varepsilon R\varepsilon v^2$.

400. Corollarium 6.

Aequatio differentio-differentialis $\overset{2}{\varepsilon}Z=M\overset{2}{\varepsilon}u+P\varepsilon u\varepsilon v+R\varepsilon v^2=0$ carens exponente differentio-differentiali $\overset{2}{\varepsilon}v$ variabilis absolutae v, et quadrato εu^2 exponentis differentialis variabilis u, erit per se primo-integrabilis, si fuerit ${}^{u}\varepsilon M=0$, et ${}^{v}\varepsilon P\varepsilon u\varepsilon v={}^{v}\overset{2}{\varepsilon}M\varepsilon u+{}^{u}\varepsilon R\varepsilon v^2$ (397. §.): debebit ergo M aut constans esse quantitas, quo casu fiet ${}^{v}\varepsilon P\varepsilon u\varepsilon v={}^{u}\varepsilon R\varepsilon v^2$, vel certa functio variabilis v, carens variabili u: omni vero casu exprimet praecedens formula (397. §.) primum illius integrale completum.

401. Corollarium 7.

Exemplo sit aequatio $\overset{2}{\varepsilon}Z=v^2(a+bv^n)\overset{2}{\varepsilon}u+v(c+ev^n)\varepsilon u\varepsilon v+(f+gv^n)u\varepsilon v^2=0$, in qua est $M=v^2(a+bv^n)$, $P=v(c+ev^n)$, $R=(f+gv^n)u$; hinc ${}^{v}\overset{2}{\varepsilon}M=(2a+(n+2)(n+1)bv^n)\varepsilon v^2$; ${}^{v}\varepsilon P=(c+(n+1)ev^n)\varepsilon v$; ${}^{u}\varepsilon R=(f+gv^n)\varepsilon u$: ut illa ergo sit per se primo-integrabilis, debet esse (400. §.) $c+(n+1)ev^n=2a+(n+2)(n+1)bv^n+f+gv^n$, unde pro casu ejusmodi integrabilitatis obtinemus $c=2a+f$, et $(n+1)e=(n+2)(n+1)b+g$.

Quapropter, nisi hae relationes inter quantitates a, b, c, e, f, g, n locum habeant, aequatio $\overset{2}{\varepsilon} Z = 0$ non erit per se integrabilis: pro his autem relationibus facile per (397. §.) definietur ejus primum integrale $\varepsilon Z = 0$, et ex hoc per praecedentia praecepta secundum (91. §.), licet aequatio $\overset{2}{\varepsilon} Z = 0$ non sit eo ipso per se etiam secundo-integrabilis. Si enim relationes inter a, b, c, e, f, g, n petas, pro quibus data aequatio sit per se primo- et secundo-integrabilis, debebis sumere aequationem $c + (n+1) e v^n = 4a + 2(n+2)(n+1) b v^n = 2f + 2g v^n$ per (399. §.), ex qua obtinentur relationes $c = 4a = 2f$, et $e = 2(n+2) b = \frac{2g}{n+1}$.

Exemplum.

Pro $n = 1$ erit aequatio $\overset{2}{\varepsilon} Z = v^2(a + bv) \overset{2}{\varepsilon} u + v(c + ev) \varepsilon u \varepsilon v + (f + gv) u \varepsilon v^2 = 0$ per se primo integrabilis, si fuerit $c = 2a + f$, et $3e = 6b + g$: posito igitur $c = 2a + f$, et $e = 3b + \frac{1}{3}g$, invenietur per (397. §.) pro primo integrali aequatio differentialis $\varepsilon Z = v^2(a + bv) \varepsilon u + v(f + \frac{1}{3}gv) u \varepsilon v + C \varepsilon v = 0$, quae non est per se integrabilis: hinc itaque elucet, assumtam aequationem $\overset{2}{\varepsilon} Z = 0$ pro illis relationibus, quae ipsam reddunt per se primo-integrabilem, non esse per se etiam secundo-integrabilem.

Ut nimirum illa sit per se primo- et secundo-integrabilis, debet esse $c = 4a = 2f$, et $e = 6b = g$: pro his autem relationibus erit per (397. §.) ejus primum integrale completum aequatio differentialis $\varepsilon Z = v^2(a + bv) \varepsilon u + v(f + \frac{1}{3}gv) u \varepsilon v + C \varepsilon v = 0$, seu $\varepsilon Z = v^2(a + bv) \varepsilon u + v(2a + 3bv) u \varepsilon v + C \varepsilon v = 0$ per (371. §.) complete integrabilis.

402. Corollarium 8.

Dum ergo integranda fuerit aequatio differentio-differentialis $\overset{2}{\varepsilon} \varphi = 0$ destituta criteriis integrabilitatis, quae adhuc exposuimus, videndum erit, annon ea certis artificiis in aequationem identicam $\overset{2}{\varepsilon} Z = 0$ possit transformari, in qua ea criteria locum habeant.

403. Problema.

Integrare aequationem differentio-differentialem $\overset{2}{\varepsilon} u + P \varepsilon u \varepsilon v + Q \varepsilon u^2 + R \varepsilon v^2 = 0$ *pro quibuscunque factoribus* P, Q, R *independentibus a variabili* u.

Solu-

Solutio.

1. Consideretur u instar functionis variabilis absolutae v, quae per datam aequationem determinata fit, quo pro certa functione Z independente a variabili u fiat $\partial u = Z\partial v$, et $\partial Z = z\partial v$, hinc $\partial^2 u = z\partial v^2$; erit factis substitutionibus $z\partial v^2 + PZ\partial v^2 + QZ^2\partial v^2 + R\partial v^2 = 0$, hinc $z = -(PZ + QZ^2 + R)$.

2. Habebimus igitur aequationem differentialem primi ordinis $\partial Z + (PZ + QZ^2 + R)\partial v = 0$ inter binas variabiles v, Z: quamobrem poterit haec aequatio saltem ope seriei cujuspiam per praecedentia praecepta integrari: invento vero hoc integrali, eruatur ex illo expressio variabilis Z per v, atque ea loco Z substituatur in $\partial u = Z\partial v$, tum hinc quaeratur u per praecepta praecedentium capitum.

404. Corollarium 1.

Hoc modo (403. §.) obtinentur sequentia integralia completa (I) (II) (III) aequationum differentio - differentialium (1) (2) (3) pro quibuslibet factoribus P, Q, R independentibus a variabili u, ubi e denotat basim logarithmorum naturalium.

1. $\partial^2 u + R\partial v^2 = 0.$ I. $u = B + Cv - \int\partial v\int R\partial v.$

2. $\partial^2 u + Q\partial u^2 = 0.$ II. $u = B + \int\frac{\partial v}{C + \int Q\partial v}.$

3. $\partial^2 u + P\partial u\partial v = 0.$ III. $u = B + \int\frac{C\partial v}{e^{\int P\partial v}}.$

405. Corollarium 2.

Eo casu determinato, quo factores P, Q, R etiam a variabili v independentes, ideoque totidem quantitates constantes fuerint; habebimus ob (403. §.) sequentes aequationes: integrale in (3) potest determinari per (253. §.); ex illo autem poterit eliei expressio variabilis Z per v, qua loco Z in (2) substituta, quaeri poterit u per praecepta praecedentium capitum.

$$1.\ \partial^2 u + P\partial u\partial v + Q\partial u^2 + R\partial v^2 = 0.$$

$$2.\ \partial u = Z\partial v. \qquad 3.\ \int\frac{\partial Z}{QZ^2 + PZ + R} = C - v.$$

406. Corollarium 3.

Quodsi in specie sit R=o vel Q=o, obtinebimus per (405. §.) sequentia integralia (I) (II) aequationum differentio-differentialium (1) (2), quidquid sint quantitates constantes P, Q, R.

$$1.\ \overset{2}{\varepsilon}u + P\varepsilon u\varepsilon v + Q\varepsilon u^2 = 0. \qquad \text{I. } u = B + \int \frac{CP\varepsilon v}{e^{Pv} - CQ}.$$

$$2.\ \overset{2}{\varepsilon}u + P\varepsilon u\varepsilon v + R\varepsilon v^2 = 0. \qquad \text{II. } u = B + \int \frac{C - Re^{Pv}}{Pe^{Pv}}\varepsilon v.$$

407. Problema.

Integrare aequationem differentio-differentialem $\overset{2}{\varepsilon}u + P\varepsilon u\varepsilon v + Q\varepsilon u^2 + R\varepsilon v^2 = 0$ *pro quibuslibet factoribus* P, Q, R *independentibus a variabili absoluta* v.

Solutio.

Ex (403. §.) obtinebimus sequentes aequationes. Cum jam factores P, Q, R careant variabili v; poterit aequatio differentialis primi ordinis in (3) per praecepta superius tradita integrari: ex integrali porro aequationis (3) poterit exprimi Z per u, et hac expressione loco Z in (2) substituta licebit demum etiam exponentem differentialem εv in (2) per regulas in praecedentibus capitibus traditas integrare, hacque ipsa integratione determinare integrale completum datae aequationis differentio-differentialis (1).

$$1.\ \overset{2}{\varepsilon}u + P\varepsilon u\varepsilon v + Q\varepsilon u^2 + R\varepsilon v^2 = 0.$$

$$2.\ \varepsilon v = \frac{\varepsilon u}{Z}. \qquad 3.\ Z\varepsilon Z + (PZ + QZ^2 + R)\varepsilon u = 0.$$

408. Corollarium 1.

Hinc (407. §.) elicientur sequentes aequationes integrales (I)(II)(III) aequationum differentio-differentialium (1) (2) (3) perfecte determinatae, quidquid sint factores P, Q, R carentes variabili v.

$$1.\ \overset{2}{\varepsilon}u + P\varepsilon u\varepsilon v = 0. \qquad \text{I. } v = B + \int \frac{\varepsilon u}{C - \int P\varepsilon u}.$$

$$2.\ \overset{2}{\varepsilon}u + R\varepsilon v^2 = 0. \qquad \text{II. } v = B + \int \frac{\varepsilon u}{\sqrt{(C - 2\int R\varepsilon u)}}.$$

$$3.\ \overset{2}{\varepsilon}u + Q\varepsilon u^2 = 0. \qquad \text{III. } v = B + \int \frac{\varepsilon u\, e^{\int Q\varepsilon u}}{C}.$$

409. C

409. Corollarium 2.

Si factores P, Q, R utraque variabili u, v careant, totidem idcirco quantitatibus constantibus aequentur; habebimus ob (407. §.) sequentes aequationes, quarum tertia per (254. §.) integrata dabit expressionem variabilis Z per u; hac vero expressione in (2) substituta reddetur exponens differentialis dv per regulas praecedentium capitum integrabilis.

$$1.\ \overset{2}{d}u + P\,du\,dv + Q\,du^2 + R\,dv^2 = 0.$$

$$2.\ dv = \frac{du}{Z}. \qquad 3.\ du = \frac{-Z\,dZ}{QZ^2 + PZ + R}.$$

410. Corollarium 3.

Bini simpliciores casus, quorum uno est $R = 0$, et altero $P = 0$, seorsim notari merentur: his enim casibus nascuntur ex (409. §.) sequentes aequationes integrales (I) (II) aequationum differentio - differentialium (1) (2) pro quibuslibet quantitatibus constantibus P, Q, R.

$$1.\ \overset{2}{d}u + P\,du\,dv + Q\,du^2 = 0. \qquad \text{I.}\ v = B + Q\int\frac{du\,e^{Qu}}{C - Pe^{Qu}}.$$

$$2.\ \overset{2}{d}u + Q\,du^2 + R\,dv^2 = 0. \qquad \text{II.}\ v = B + \sqrt{Q}\int\frac{du\,e}{\sqrt{(C - Re^{2Qu})}}.$$

Scholion 1.

Nonnunquam poterit integratio aequationis differentio - differentialis reduci ad integrationem unius tantum aequationis differentialis primi ordinis, si aut primi exponentes differentiales du, dv, variabilium u, v, a quibus data aequatio dependet, aut certae variabilium u, v, vel etiam exponentium differentialium du, dv functiones instar quantitatum variabilium considerentur, atque hae novis characteribus expressae in data aequatione substituantur.

E. gr. Si in $\overset{2}{d}y = du\,dv + u\,\overset{2}{d}v$ fiat $dv = z$, hinc $\overset{2}{d}v = dz$; erit $\overset{2}{d}y = z\,du + u\,dz$: igitur $dy = zu = u\,dv$. In $\overset{2}{d}y = dv\,\overset{2}{d}u + du\,\overset{2}{d}v$ ponatur $dv = x$, $du = z$; erit $\overset{2}{d}y = x\,dz + z\,dx$: igitur $dy = zx = du\,dv$.

In $dV = \frac{2u^2\,dv\,\overset{2}{d}v - 2u\,du\,dv^2}{u^4}$ fiat $u^2 = z$ et $dv^2 = x$; erit $2u\,du = dz$, $2dv\,\overset{2}{d}v = dx$: igitur $dV = \frac{z\,dx - x\,dz}{z^2}$, et $V = \frac{x}{z} = \frac{dv^2}{u^2}$.

Scho-

Scholion 2.

Notet autem sibi tyro, multum esse in eo positum, ut variabilis absoluta noscatur, ad quam aequatio integranda refertur, quod quidem in adplicatione calculi differentio-integralis ad resolutionem problematum mathematicorum nullam patitur difficultatem.

E. gr. Si proponatur aequatio $Ay^m \partial y \partial z = y \partial^2 x + \partial x \partial y$, ea haud facile integrabitur, nisi praevie scias, variabilem z absolutam, adeoque exponentem differentialem $\partial z = 1$ constantem esse: in hac vero hypothesi erit posito $\partial x = v$, integrale $\int Ay^m \partial y \partial z = \int (y \partial v + v \partial y)$, adeoque $\frac{Ay^{m+1} \partial z}{m+1} = yv + C \partial z = y \partial x + C \partial z$.

Si in resolutione cujuspiam problematis deveniatur ad aequationem $V \partial y = \frac{\partial y \partial x^2 - y \partial y \partial^2 y}{y^3 \partial x^2}$, non adparebit, qua ratione illius integrale possit determinari, nisi constet, variabilem $u = \int y \partial x$ pro absoluta, adeoque exponentem differentialem $\partial u = y \partial x$ pro constanti fuisse sumtum; quo casu erit utique etiam $\partial u^2 = y^2 \partial x^2$ constans, et ideo, cum sit $V \partial y = \frac{\partial y}{y^3} - \frac{\partial y \partial^2 y}{y^2 \partial x^2}$, habebitur $\int V \partial y = \int \frac{\partial y}{y^3} - \frac{1}{y^2 \partial x^2} \int \partial y \partial^2 y = C - \frac{1}{2y^2} - \frac{\partial y^2}{2y^2 \partial x^2}$.

Scholion 3.

Consideratio haec variabilis absolutae, cujus primus exponens differentialis $= 1$ constans sit, maximi est momenti in negotio integrationis aequationum differentio-differentialium. Frequenter sane fiet, ut proposita aequatio differentio-differentialis dependens a quapiam variabili absoluta, quae nullo modo videtur esse integrabilis, facile integretur, si alia aliqua variabilis pro absoluta sumatur. Verum, ut haec nova variabilis absoluta rite possit determinari, necesse est, ut ante omnia data aequatio differentio-differentialis in aliam transformetur, in qua nulla variabilis sumitur pro absoluta, quod hoc modo praestabitur.

1. Data aequatio complectatur $\partial^2 u$, ∂u, ∂v, denotante v variabilem absolutam, et $\partial v = 1$ exponentem differentialem constantem. Cum in hac hypothesi spectetur u instar certae functionis variabilis v; debet esse $\partial u = p \partial v$, et $\partial p = q \partial v$ pro quibusdam functionibus p, q variabilis v: quare, dum

dum v pro variabili absoluta sumitur, est $\overset{2}{\partial}u = q\partial v^2$; si autem v non sumeretur pro variabili absoluta, fieret $\overset{2}{\partial}u = \partial p\partial v + p\overset{2}{\partial}v = q\partial v^2 + \frac{\partial u\overset{2}{\partial}v}{\partial v}$.

2. Hinc (1) sequitur, transformationem aequationis differentio-differentialis, dependentis a variabili absoluta v et a certa functione u ejus variabilis, substitutione functionis $\frac{\partial v\overset{2}{\partial}u - \partial u\overset{2}{\partial}v}{\partial v} = \overset{2}{\partial}u - \frac{\partial u\overset{2}{\partial}v}{\partial v}$ loco $\overset{2}{\partial}u$ absolvi.

Scholion 4.

Quod porro ad delectum variabilis pro absoluta sumendae attinet, quo integratio datae aequationis differentio-differentialis facilior reddatur, is ingenio ejus, qui integrationem debuerit perficere, relinquendus est, cum nullum generale criterium in eum finem liceat assignare. Notet tamen sibi tyro, aequatione differentio-differentiali per (2. Schol.) transformata non facile alium exponentem differentialem pro constanti esse sumendum, quam cujus integratio est in potestate, cum hac ipsa integratione, quantitas variabilis pro absoluta sumenda debeat determinari.

E. gr. detur integranda aequatio $V\partial v = \frac{\partial v\partial u^3 + \partial u\partial v^3 + v\partial v^2\overset{2}{\partial}u}{2v^3\partial u^3}$, in qua est v variabilis absoluta, et $\partial v = 1$ constans: facta transformatione per (2. Schol.) obtinebitur aequatio $V\partial v = \frac{\partial v\partial u^3 + \partial u\partial v^3 + v\partial v^2\overset{2}{\partial}u - v\partial u\overset{2}{\partial}v\partial v}{2v^3\partial u^3}$, in qua nulla variabilis est absoluta. Quodsi jam pars $\partial u\partial v^3 + v\partial v^2\overset{2}{\partial}u = \partial v^2(\partial u\partial v + v\overset{2}{\partial}u)$ numeratoris seorsim consideretur, illico adparet, non modo $v\partial u$ esse perfectum integrale factoris $\partial u\partial v + v\overset{2}{\partial}u$, sed etiam hunc ipsum factorem fieri aequalem nihilo, si ejus integrale $v\partial u$ pro constanti sumatur. Quamobrem, quo simplicior reddatur expressio aequationis integrandae, sumatur $v\partial u$ pro constanti, seu $\int v\partial u$ pro variabili absoluta: erit enim $\partial(v\partial u) = \partial v\partial u + v\overset{2}{\partial}u = 0$, et $V\partial v = \frac{\partial v\partial u^3 - v\partial u\overset{2}{\partial}v\partial v}{2v^3\partial u^3}$. Hinc, quia $v\partial u$, adeoque etiam $2v^2\partial u^2$ constans est, fiet integrale $\int V\partial v = \frac{1}{2}\int\frac{\partial v}{v^3} - \frac{1}{2v^2\partial u^2}\int\partial v\overset{2}{\partial}v = C - \frac{1}{4v^2} - \frac{\partial v^2}{4v^2\partial u^2}$.

Scholion

Scholion 4.

Similibus methodis solet peragi integratio aequationum differentialium altiorum ordinum: quaevis enim aequatio differentialis ordinis $n+1$ exprimit ejusmodi aequationem, quae per differentiationem certae aequationis differentialis ordinis $n$ti obtenta potest censeri, quae idcirco id est respectu hujus aequationis, quod aequatio differentio-differentialis, seu aequatio differentialis secundi ordinis respectu aequationis differentialis primi ordinis. Casum simplicissimum exhibet aequatio formae $\overset{n}{d}u = V dv^n$, in qua v est variabilis absoluta, hinc $dv = 1$ constans integratio ejusmodi aequationis facile perficitur, quaerendo successive exponentes differentiales $\overset{n-1}{d}u$, $\overset{n-2}{d}u$, $\overset{n-3}{d}u$, - - - - - du functionis u, ex quorum ultima determinatur ipsa functio u. Si esset $\overset{r}{d}u = p\,dv^r$, cum sit dv constans, et $\overset{r}{d}u = d.\overset{r-1}{d}u$; deberet utique fieri $\int \overset{r}{d}u = \overset{r-1}{d}u$; et $\int p\,dv^r = dv^{r-1}\int p\,dv = (P+C)\,dv^{r-1}$, posito $\int p\,dv = P$: igitur esset $\overset{r-1}{d}u = (P+C)\,dv^{r-1}$, unde manifestum fit, qua ratione ex $\overset{n}{d}u = V dv^n$ exponentes differentiales $\overset{n-1}{d}u$, $\overset{n-2}{d}u$, - - - du successive possint determinari: erit nimirum $\overset{n-1}{d}u = (\alpha + A)\,dv^{n-1}$ pro $\alpha = \int V dv$ et constante indeterminata A: $\overset{n-2}{d}u = dv^{n-2}\int(\alpha dv + A dv) = (\beta + Av + B)\,dv^{n-2}$ pro $\beta = \int \alpha dv$, et constante indeterminata B: $\overset{n-3}{d}u = dv^{n-3}\int(\beta dv + Av dv + B dv) = (\gamma + \frac{1}{2}Av^2 + Bv + C)\,dv^{n-3}$, pro $\gamma = \int \beta dv$ et constante indeterminata C: et ita porro.

ELEMEN-

ELEMENTA
GEOMETRIAE SUBLIMIORIS.

CAPUT VIII.

DE GENERALIBUS QUIBUSDAM CURVARUM PROPRIETATIBUS.

411. Definitio.

Generalis omnium curvarum diviſio eſt in *curvas ſimplicis et duplicis curvaturae*: omnis curva, quae tota jacet in uno plano, vocatur curva *ſimplicis curvaturae:* omnes vero reliquae curvae ſunt *curvae duplicis curvaturae.* Interea de curvis tantum ſimplicis curvaturae tractabimus.

412. Definitio.

Si in eodem plano dentur curva ZMPX (2. Fig.) et binae rectae Fig. 2.
VU, DE in puncto A ſeſe interſecantes, atque ex A in recta VU abſcindantur quaecunque partes AQ, AN, tum ex punctis Q, N ducantur rectae QP, NM parallelae alteri rectae DE; vocabuntur AQ, AN *Abſciſſae*, et QP, NM iis reſpondentes *Ordinatae* ſeu *Applicatae;* punctum A erit *Origo* ſeu *Initium abſciſſarum;* recta vero VU appellabitur *Linea* ſeu *Axis abſciſſarum*, et recta DE *Axis ordinatarum.*

413. Corollarium.

Situs abſciſſae AN reſpectu originis A eſt ſitui abſciſſae AQ, et ſitus ordinatae QP reſpectu lineae abſciſſarum VU ſitui ordinatae NM e diametro oppoſitus: eapropter, dum in calculis abſciſſae AQ et ordinatae NM ſpectantur ut *poſitivae*, ſolent abſciſſae AN et ordinatae QP prioribus oppoſitae haberi pro *negativis.*

414. Definitio.

Quaevis abſciſſa AQ et ordinata QP ſimul vocantur *Coordinatae:* hinc AQP eſt *Angulus coordinatarum* aequalis angulo DAV, ſub quo *axes coordinatarum* VU, DE ſeſe interſecant. Sunt autem coordinatae AQ, QP *orthogonae* ſeu *rectangulae*, vel *obliquangulae*, prout illarum angulus AQP eſt rectus, vel major aut minor recto.

415. Definitio.

Fig. 3. Recta ab *tanget* datum alicujus curvae arcum MOX (3. Fig.) in puncto M, si ea arcui MOX ita occurrat in puncto M, ut inter ipsam et arcum nulla recta per idem punctum M possit transire; hinc vocabitur M *Punctum contactus.*

416. Corollarium 1.

Inter innumeras rectas, quae dato arcui MOX in puncto M possunt occurrere, unica tangit arcum in M: possibile quoque est, ut aliqua recta cd arcum MO curvae MOX tangat in puncto O, quin ideo etiam arcus OX ab eadem recta tangatur in O. Quamobrem dicetur curva MOX a recta cd tangi in puncto intermedio O, si recta cd utrumque arcum OM, OX adjacentem puncto O tangat in O, arcusque hi ad eandem partem illius Fig. 4. rectae jaceant: secus enim recta cd (4. Fig.) secabit curvam MOX, licet fors uterque illius arcus MO, XO ab ea in O tangatur.

417. Corollarium 2.

Fig. 5. 6. 7. 8. Si, sumta in puncto A rectae VU (5. 6. 7. 8. Fig.) origine abscissarum, ductaque ad punctum M, in quo recta TS arcum MX tangat, ordinata orthogona MP, abscissa AP capiat incrementum PQ, novaeque abscissae AQ respondeat ordinata orthogona Qm; poterit per puncta M, m duci recta RL, quae cum tangente TS angulum LMS=RMT ad punctum contactus M formet: decrescente PQ, decrescet angulus SML=RMT, poteritque is fieri minor dato quovis angulo (415. §.) Cum igitur angulus SML=RMT aequetur differentiae angulorum RMP, TMP, quos secans RM, et tangens TM cum ordinata orthogona MP in puncto contactus M formant; evidens est, situm tangentis TM respectu ordinatae orthogonae MP ita esse determinatum, ut angulus TMP inter tangentem et ordinatam in puncto contactus aequetur limiti, ad quem, PQ continuo decrescente, angulus RMP inter secantem et ordinatam eo propius accedat, quo magis minuitur PQ (68. §.).

418. Corollarium 3.

Ex his manifestum fit, rectam, quae datum arcum in aliquo puncto M tangit, non posse cum linea abscissarum VU concurrere, quin punctum concursus T ad eandem partem ordinatae orthogonae MP, ad quam jacet origo abscissarum A, vel ad partem oppositam cadat, prout crescentibus abscissis Fig. 5. 6. 7. 8. AP ordinatae crescunt (5. 6. Fig.), vel decrescunt (7. 8. Fig.)

419. De-

419. Definitio.

Si recta TS arcum MX tangat in puncto M, lineae vero abscissarum VU occurrat in T, ducaturque ad punctum contactus M ordinata MP, et recta MN priori TS perpendicularis, atque lineae abscissarum occurrens in N; erit TM *Tangens* determinatae longitudinis; TP *Subtangens*; MN *Normalis*; et NP *Subnormalis*.

420. Corollarium 1.

Tangens cum subtangente, et normalis cum subnormali cadunt ad partes oppositas ordinatae orthogonae MP.

421. Corollarium 2.

Datis ordinata orthogona MP et angulo TMP inter tangentem TM et ordinatam MP in puncto contactus M, poterit determinari tangens $TM = \frac{MP}{\text{Cof}\,TMP}$; subtangens $TP = MP . \text{Tang}\,TMP$; normalis $MN = \frac{MP}{\text{Sin}\,TMP}$; et subnormalis $NP = MP . \text{Cot}\,TMP$.

422. Corollarium 3.

Si rectae cd, ab (3. Fig.) arcum MO datae curvae tangant in punctis M, O, ducanturque normales ML, ON, et chorda MO; erunt anguli cOM, aMO, LMO, NOM acuti: igitur secabunt sese tangentes cd, ab alicubi in e, et normales alicubi in n. Fig. 3.

423. Definitio.

Angulus MnO, sub quo normales ML, ON punctis M, O dati arcus MO alicujus curvae respondentes sese intersecant (422. §.), solet vocari *Amplitudo* arcus MO: angulus vero ceb, sub quo tangentes cd, ab ductae ad puncta O, M sese secant (422. §.), est Angulus *inclinationis* unius tangentis ad alteram.

424. Corollarium 1.

Amplitudo MnO cujusvis arcus MoO aequatur angulo inclinationis ceb tangentis cO ad tangentem aM: quare, cum arcus MO post punctum contactus M versus O eo magis deflectat a tangente Ma, isque idcirco eo magis curvetur in M (415. §.), quo major est angulus ceM; clarum est, curvedinem datae curvae in puncto M ita esse connexam cum amplitudine

arcus MoO, ut illa eo major debeat censeri, quae major est amplitudo MnO arcus MO determinatae longitudinis, et vicissim.

425. Corollarium 2.

Fig. 1. In circulo (1. Fig.) radii $r = AC$ erunt pro quolibet arcu PD normales punctis P, D respondentes, ipsi radii PC, DC (419. §.): quodsi ergo sit arcus $PD = A$; erit angulus $PCD = \frac{A}{rr}$ (5. §. 3. Schol.) amplitudo arcus $PD = A$ (423. §.).

426. Corollarium 3.

Ubicunque in dato circulo datisque circulis aequalibus sumantur arcus aequales utcunque parvi; erunt amplitudines omnium arcuum inter se aequales (425. §.): curvedo igitur in quovis circulo circulisque aequalibus constans est, nimirum eadem in singulis punctis peripheriae unius circuli, peripheriarumque circulorum aequalium (424. §.).

427. Corollarium 4.

Si vero duo circuli inaequales describantur radiis r, R; erit amplitudo cujuslibet quantumcunque parvi arcus A in circulo radii r ad amplitudinem arcus aequalis A in circulo radii R, sicut est R ad r (425. §.): quamobrem, si curvedines constantes duorum circulorum inaequalium (426. §.) inter se comparentur; debebit esse curvedo in circulo radii minoris major quam in circulo radii majoris (424. §.).

428. Definitio.

Fig. 3. *Circulus curvedinis* pro quocunque puncto O datae curvae ZMOX (3. Fig.) vocabitur circulus, ad cujus constantem curvedinem (426. §.) curvedo curvae ZMOX in puncto O proxime accesserit, ita nimirum, ut differentia inter hanc et illam curvedinem minor sit, quam esset inter eandem curvae ZMOX curvedinem et curvedinem constantem cujuscunque alterius majoris minorisque circuli (427. §.): radius centrumque circuli curvedinis pro puncto O datae curvae brevitatis causa appellabitur *Radius centrumque curvedinis* datae curvae in puncto O.

429. Corollarium 1.

Cognitis radiis curvedinum in duobus punctis unius curvae, aut duarum diversarum curvarum; patebit eo ipso, in quonam puncto major, in quove minor curvedo est: major erit, cujus radius minor fuerit (428. 427. §.)

430. Co-

430. Corollarium 2.

Quodsi ergo a puncto O versus Z sumantur quaecunquae datae curvae puncta, noscanturque radii curvedinum in singulis punctis; constabit eo ipso, an curvatura a puncto O versus Z crescat continuo vel decrescat, aut alterne crescat et decrescat; et an eadem sit in punctis a dato puncto aequidistantibus (429. §.).

431. Corollarium 3.

Si in normali ON ducta ad punctum contactus O sumatur punctum t, et radio tO describi cogitetur circulus Oα cadens intra arcum OX datae curvae, ducaturque ms tangenti cd parallela, adeoque normali ON perpendicularis; erit cd communis tangens arcus OX et circuli Oα: differentia vero inter curvaturam curvae ZOX in puncto O versus X et curvedinem constantem circuli Oα erit eo minor, quo minor fuerit differentia inter deflexus arcus OX circulique Oα a communi tangente Od in regione puncti contactus O, quo minor idcirco fuerit differentia mr inter ordinatas orthogonas ms, rs arcus OX et circuli Oα eidem quantumcunque parvae abscissae Os a puncto contactus computatae respondentes. Quapropter, si constiterit differentiam mr ejusmodi ordinatarum minorem esse, quam esset differentia inter ordinatam ms arcus OX, et ordinatam ks vel us cujuscunque alterius circuli Oβ vel Oγ, qui majori radio gO, vel minori fO descriptus concipiatur; constabit eo ipso, circulum Oα esse circulum curvedinis in puncto O datae curvae (428. §.); et vicissim.

432. Corollarium 4.

Sic quoque, si radio tO descriptus circulus Oβ extra arcum OX cadat; erit is, ob pares rationes, circulus curvedinis in puncto O datae curvae, si differentia inter ordinatam ms arcus OX et ordinatam ks circuli Oβ eidem quantumcunque parvae abscissae Os respondentem minor fuerit, quam esset differentia inter ordinatam ms arcus OX, et ordinatam rs vel vs circuli Oα vel Oδ quocunque alio radio minori fO, vel majori gO descripti; et vicissim.

433. Corollarium 5.

Quiscunque idcirco circulus radio tO descriptus fuerit circulus curvedinis in puncto O; debebit is habere arcum in regione puncti contactus O adeo vicinum arcui OX datae curvae, ut inter illum et hunc

arcum

arcum nullus circulus possit transire, seu is minori radio fO, seu majori gO describi cogitetur (431. 432. §.): ob hanc causam dicetur circulus curvedinis datae curvae in puncto O versus X *osculari* arcum OX; hinc appellabitur is *circulus osculator*; ejusque radius tO *radius osculi*, vel *radius circuli osculatoris*.

434. Corollarium 6.

Circulus radio tO descriptus arcumque OX datae curvae in O osculans tangit eundem arcum in puncto O interne vel externe (433. §.).

435. Theorema.

Si in data curva MOX *fuerit summa quadratorum ordinatae* ms *et abscissae* Os *divisa per duplam abscissam* Os *aequalis summae* $Z+\psi$ *vel differentiae* $Z-\psi$ *duarum quantitatum* Z, ψ, *quarum prior independens sit ab abscissa* Os, *posterior vero ita ab ea pendeat, ut decrescente abscissa* Os *possit* ψ *fieri minor data quavis quantitate; erit* $Z=tO$ *radius circuli curvedinis in puncto* O, *qui arcum* OX *tanget in puncto* O *interne casu primo, et externe casu secundo.*

Demonstratio.

1. Per hypothesim erit $\frac{ms^2+u^2}{2u}=Z+\psi$, hinc $ms^2=2Zu-u^2+2\psi u$ pro abscissa $Os=u$.

2. Vel $\frac{ms^2+u^2}{2u}=Z-\psi$, hinc $ms^2=2uZ-u^2-2\psi u$.

3. Quodsi jam radio $tO=Z$ cogitetur circulus esse descriptus, in quo abscissae $u=Os$ respondeat ordinata $y=rs$, vel $y=ks$; erit $y^2=2Zu-u^2$.

4. Hinc jam patet evidenter, semper esse $ms>y$ in prima hypothesi (1), et $ms<y$ in secunda hypothesi (2): in prima hypothesi jacebit ergo arcus circularis radio $tO=Z$ descriptus alicubi in $O\alpha$ intra arcum OX datae curvae; in secunda vero hypothesi jacebit is alicubi in $O\beta$ extra arcum OX.

5. Ponamus jam in prima hypothesi (1) praeter circulum $O\alpha$ radio $tO=Z$ descriptum describi circulum $O\beta$ radio $gO=Z+\omega$, pro differentia $\omega=gt$ quantumcunque parva; respondebit in hoc circulo abscissae $Os=u$ quadratum ordinatae $ks^2=2Zu+2\omega u-u^2$.

6. Cum

6. Cum per hypothesim possit fieri $\psi < \omega$ decrescente abscissa $u = Os$; extabunt abscissae $Os = u$ pro quibus erit $ms < ks$ in (1) (5), unde necessario sequitur, arcum circularem $O\beta$ radio $Og > Z$ descriptum in regione puncti contactus O cadere extra arcum OX datae curvae; arcum proinde $O\alpha$ radio $tO = Z$ descriptum in regione puncti contactus O adeo esse vicinum arcui OX, ut inter hunc et illum nullus alter circulus locum habeat. Verum, ut certo constet, hunc ipsum circulum $O\alpha$ esse circulum curvedinis in puncto O, demonstrari adhuc debet, extare abscissam $u = Os$, pro qua quantumcunque decrescente sit $ms - rs < ks - ms$ (431. §.), quod hoc modo potest ostendi.

Per hypothesim extabit abscissa $u = Os$, pro qua, quantumcunque ea decrescat, erit $\psi < \frac{\omega}{2}$: igitur, sumtis pro ms, rs, ks valoribus juxta (1) (3) (5) erit etiam $rs^2 + 2ks.rs + ks^2 > 4ms^2$, hinc $rs + ks > 2ms$, et ideo etiam $ks - ms > ms - rs$.

7. In secunda hypothesi (2), descripto circulo $O\beta$ radio $tO = Z$ ob (4), describatur circulus $O\gamma$ radio minori $Of = Ot - ft = Z - \omega$, sumta quantumcunque exigua differentia $\omega = tf$; respondebit in hoc circulo abscissae $Os = u$ quadratum ordinatae $us^2 = 2Zu - 2\omega u - u^2$.

8. Hinc, quia per hypothesim potest, decrescente abscissa $u = Os$, fieri $\psi < \omega$, manifestum fit, extare debere abscissas u, pro quibus quantumcunque decrescentibus fieret $us^2 < ms^2$ et $us < ms$ in (7)(2); consequenter omnem circulum radio $fO < Z$ descriptum jacere in regione puncti contactus O intra arcum OX datae curvae; et ideo circulum $O\beta$ radio $tO = Z$ extra arcum OX descriptum (4) adeo esse vicinum arcui OX in regione puncti contactus O, ut inter illum et hunc arcum nullus circulus possit cadere. Quamobrem, quo certo constet, circulum $O\beta$ fore circulum curvedinis in puncto O, ostendi adhuc debet, dari abscissas $u = Os$ in secunda hypothesi (2), pro quibus, quantumcunque eae decrescant, semper sit $ks - ms < ms - us$.

Cum enim per hypothesim possit fieri $\psi < \frac{\omega}{2}$; erit, sumtis pro ks, ms, us valoribus ex (3)(2)(7), etiam $ks^2 + 2ks.us + us^2 < 4ms^2$, hinc $ks + us < 2ms$, et ideo $ks - ms < ms - us$.

436. Corollarium.

Cum in regione puncti contactus O extent abscissae $u = Os$, pro quibus quantumcunque decrescentibus semper sit $ms > rs$, vel $ms < ks$ pro circulo curvedinis $O\alpha$ vel $O\beta$ radio $tO = Z$ descripto, ob (n. 4); patet evidenter, circulum curvedinis $O\alpha$, vel $O\beta$ radio $tO = Z$ descriptum nullum absolute arcum habere, quantumcunque parvus is concipiatur, qui in regione puncti contactus O cum certa parte arcus OX datae curvae coincidat, licet is huic arcui adeo sit vicinus, ut inter illum et hunc nullus arcus circuli possit jacere: circulus idcirco curvedinis in puncto O eam habet naturam, ut curvedo datae curvae in puncto O versus X non sit eadem cum constanti curvedine ejus circuli, sed ad eam tantum proxime accedat, ita nimirum ut illa ad nullius alterius circuli curvedinem propius possit accedere (428. §.).

437. Definitio.

Discrimen generale inter *Ramos* curvarum in eo consistit, quod aliqui sint *definitae*, alii autem *indefinitae longitudinis*, nimirum respectu datarum linearum, rectarum vel curvarum, juxta quas excurrunt. Sic ramus PX

Fig. 2. (2. Fig.) dicetur *in indefinitum excurrere juxta datam rectam* VU, si is, quantumcunque producatur versus U, nuspiam concurrat cum recta VU: vocabitur porro recta VU *Asymptotus* rami PX, si hic juxta rectam VU in indefinitum excurrens eo propius ad illam accedat, quo magis versus U producitur. Hoc modo solent concipi rami curvarum juxta ramos aliarum curvarum in indefinitum excurrentes, asymptotique curvilineae: verum de his nullibi erit sermo in sequentibus.

438. Definitio.

Possibile est, ut uni eidemque abscissae in data curva una tantum ordinata, positiva vel negativa, aut plures ordinatae, omnes positivae, vel omnes negativae, aut aliquae positivae et aliae negativae, illis numero

Fig. 9. aequales, vel inaequales (413. §.) respondeant. Sic e. gr. in (9. Fig.) abscissae At una respondet ordinata, positiva ts; abscissae Ar respondent duae ordinatae, positivae sp, rq; abscissae An respondent pariter binae ordinatae, una positiva nm, et altera negativa no; abscissae vero Ak respondent duae ordinatae positivae kh, ki, et una negativa kl; et abscissae Ae respondent quatuor ordinatae positivae, ea, eb, ec, ed, et duae negativae ef, eg.

Omnis

Omnis linea abſciſſarum vocatur generatim *Diameter* curvae, ſi ſumma ordinatarum poſitivarum cuivis abſciſſae reſpondentium aequetur ſummae ordinatarum negativarum reſpondentium eidem abſciſſae. Eo autem determinato caſu, quo ad extremitatem cujusvis abſciſſae cuilibet ordinatae poſitivae reſpondet una ordinata negativa priori aequalis; appellatur ea *Diameter abſoluta.* Diameter porro abſoluta, ad quam ordinatae ſunt perpendiculares, vocari ſolet etiam *Axis curvae;* et punctum, in quo curva ab ejus axe ſecatur, eſt *Vertex* curvae. Si denique aliqua curva binas diametros habeat, quarum quaevis ſit parallela ordinatis ad aliam diametrum; erunt eae *Diametri conjugatae.*

439. Definitio.

Centrum curvae eſt punctum, in quo omnis curvae chorda per ipſum tranſiens biſſecatur.

440. Definitio.

Dum plures alicujus curvae rami in uno puncto concurrunt, ſeſe tangunt, aut interſecant, dicitur id, improprio quidem ſenſu, ex pluribus ejusdem curvae punctis *coaleſcere*, quae nimirum totidem ejus ramis debentur. Punctum curvae, quod ex pluribus ejusdem curvae punctis non coaluit, vocatur *ſimplex;* omne vero, quod ex pluribus punctis coaluit, dicitur *multiplex*, atque in ſpecie *duplex, triplex*, et ſic porro.

441. Definitio.

Ad puncta multiplicia pertinent puncta *flexus contrarii; puncta regreſſus*, ſeu *cuſpides* aut *puncta reflexionis;* et *nodi.* Si duo arcus MO, Fig. 4.
OX alicujus curvae ita concurrant in puncto O, ut recta cd, quae curvam in O ſecat, utrumque arcum MO, XO in O tangat, et, dum unus arcus MO parti cO communis tangentis cd obvertit convexitatem, eam alter arcus OX obvertat parti oppoſitae Od tangentis; vocabitur O *punctum flexus contrarii.* Quodſi autem uterque arcus eidem parti cO communis tangentis cd convexitatem obvertat (10. 11. Fig.); erit punctum duplex O Fig. 10. 11.
Cuſpis curvae, ſeu *Punctum regreſſus.* Puncta denique, in quibus arcus curvae ſeſe, ut in α, β (9. Fig.) interſecant, appellantur *Nodi.*

442. Definitio.

Aequatio ad curvam eſt aequatio eam relationem inter curvae coordinatas, vel quaspiam rectas lineas anguloſque variabiles, et conſtantes

quantitates (lineas vel angulos) exprimens, per quam natura curvae perfecte determinatur. Hinc desumitur divisio curvarum in *algebraicas* et *transcendentes*, prout aequationes ad illas sunt algebraicae vel transcendentes (160. §.). Curvae porro algebraicae dividuntur in diversos *Ordines*, ita ut omnis curva sit generatim *nti Ordinis*, si aequatio ad illam est *nti* ordinis (164. §.). Omnis demum curva *continua* est, seu *regularis*, vel *discontinua* seu *irregularis*: curva *continua* vocatur, cujus natura per unicam aequationem exprimitur: *discontinuas* vero curvae, quales in sequentibus haud considerabimus, appellantur, quarum variae partes per varias aequationes debent exprimi.

443. Corollarium 1.

Formula generalis omnium aequationum ad curvas inter binas quascunque variabiles quantitates x, y potest esse $\alpha + \overset{x}{\varphi} + \overset{y}{\mu} + \overset{xy}{\psi} = 0$, si α denotet quantitatem constantem, $\overset{x}{\varphi}$ vero et $\overset{y}{\mu}$ sint summae omnium terminorum, in quibus sola variabilis x, vel sola y continetur, et $\overset{xy}{\psi}$ exprimat summam reliquorum terminorum, utramque variabilem x, y complectentium (442. §.).

444. Corollarium 2.

Quaevis variabilium y, x contentarum in data aequatione ad curvam, puta y, potest spectari instar certae functionis alterius variabilis x, cujus exponentes differentiales εy, $\overset{2}{\varepsilon} y$, $\overset{3}{\varepsilon} y$, etc., ex data aequatione possint elici: primus vero exponens differentialis εy, prouti is positivi vel negativi valoris fuerit pro certa abscissa x, denotabit, ordinatas y, crescentibus abscissis, crescere vel decrescere (85. §.).

445. Problema.

Fig. 12. *Data aequatione ad curvam* ZMX (12. Fig.) *inter ejus coordinatas orthogonas* x = AP, y = MP, *invenire aequationem ad eandem curvam inter ipsius coordinatas obliquangulas* u = AQ, z = MQ, *cognito angulo* φ = MQA *harum coordinatarum.*

Solutio.

Cum debeat esse $y = z \operatorname{Sin} \varphi$ et $x = u - z \operatorname{Cos} \varphi$, si hos valores loco y, x substituas in data aequatione inter x, y, obtinebis aliam aequationem

sequi-

aequivalentem inter u et z: semper vero erit $\text{Cos}\,\phi = \text{Cos}\,MQA$ sumendus cum signo —, ubi angulus $\phi = MQA$ versus originem abscissarum A fuerit obtusus.

446. Corollarium 1.

Eadem prorsus ratione, si determinata sit distantia $QA = a$ puncti Q ab origine abscissarum A, ponaturque $y = MP$, $x = AP$, $MQ = z$, $MQA = \gamma$, obtinebuntur valores $y = z\,\text{Sin}\,\gamma$, $x = a - z\,\text{Cos}\,\gamma$, pro quibus loco y, x in data ad curvam ZX aequatione inter ejus ordinatas orthogonas x, y, producetur aequatio ad eandem curvam inter ordinatas $z = MQ$ sub angulis variabilibus $\gamma = MQA$ ad datum in linea abscissarum VU punctum Q *convergentes*, et ipsos angulos γ.

447. Corollarium. 2.

Commodissime exploratur natura curvae ope coordinatarum orthogonarum; deinceps idcirco omnes disquisitiones ad orthogonas coordinatas restringemus, eas ubique, nisi contrarium notetur, nomine ordinatarum intellecturi: facile enim, ubi opus fuerit, quasvis expressiones analyticas, complectentes coordinatas orthogonas certae curvae, in alias aequivalentes, quae coordinatas obliquangulas ejusdem curvae complectantur, licebit convertere (445. §.).

448. Problema.

Data aequatione ad curvam RTMS (13. Fig.) *inter ordinatas* $x = AP$, $y = MP$ *pro axe abscissarum* VX, *invenire aequationem ad eandem curvam inter ordinatas* $u = NQ$, $z = MQ$ *pro alio axe* Uu, *cujus situs respectu axeos* VX *ita sit determinatus, ut noscantur, praeter angulum* $XHu = \alpha$, *distantiae perpendiculares* $NL = b$, $Nn' = a$ *novae originis abscissarum* N *ab axibus* VX, DE. Fig. 13.

Solutio.

1. Ductis perpendicularibus Pq, Pr ad MQ, Uu, invenietur $Pq = y\,\text{Sin}\,\alpha$; $Mq = y\,\text{Cos}\,\alpha$; $HL = \frac{b\,\text{Cos}\,\alpha}{\text{Sin}\,\alpha}$; $HN = \frac{b}{\text{Sin}\,\alpha}$.

2. Est vero $Pr = HP.\text{Sin}\,PHr = (HL + LA + AP)\,\text{Sin}\,\alpha$, et $Hr = HP.\text{Cos}\,PHr$: per (1) fiet ergo $Pr = b\,\text{Cos}\,\alpha + a\,\text{Sin}\,\alpha + x\,\text{Sin}\,\alpha$; $Hr = \frac{b\,\text{Cos}\,\alpha^2}{\text{Sin}\,\alpha} + a\,\text{Cos}\,\alpha + x\,\text{Cos}\,\alpha$.

 3. De-

3. Debet porro esse $u = NQ = Hr - Pq - HN$, et $z = MQ = Mq + Pr$: igitur fiet per (2) $u = (a+x)\,Cos\,\alpha - (y+b)\,Sin\,\alpha$; $z = (y+b)\,Cos\,\alpha + (a+x)\,Sin\,\alpha$.

4. Ex his demum aequationibus prodibunt sequentes valores pro y, x, quos in data aequatione oportebit substituere, ut ea det aequationem inter coordinatas u, z.

$$y = z\,Cos\,\alpha - u\,Sin\,\alpha - b. \qquad x = z\,Sin\,\alpha + u\,Cos\,\alpha - a.$$

449. Corollarium 1.

Quaecunque recta sumatur pro axe abscissarum datae curvae, ejusdem semper ordinis erit aequatio inter coordinatas orthogonas ejusdem curvae (448. 442. §.).

450. Corollarium 2.

Solutio praecedentis problematis ita est comparata, ut ea etiam ad casum lineae rectae RTMS possit extendi, quae, si illa lineam abscissarum VX secet sub angulo $\varphi = STX$ in T, et axem ordinatarum DE in n, sitque $An = e$, $mn = AP = x$, dabit aequationem $y = e + x\,Tang\,\varphi$ primi ordinis: aequatio igitur ad lineam rectam RTnS erit primi ordinis pro quovis possibili axe abscissarum (449. §.).

451. Corollarium 3.

Quidquid sint coefficientes α, β, γ in aequatione generali $\alpha + \beta x + \gamma y = 0$ primi ordinis, poterit, ductis axibus coordinatarum VX, DE, sumi $An = -\frac{\alpha}{\gamma}$, et $AT = \frac{\alpha}{\beta}$: quodsi ergo ducatur per puncta n, T recta RTnS; debebit pro qualibet abscissa $x = AP$ esse ordinata $y = MP = An + Mm = -\frac{\alpha}{\gamma} - \frac{\beta x}{\gamma}$, hinc $\alpha + \beta x + \gamma y = 0$. Quare, sicut aequatio ad lineam rectam semper est primi ordinis (450. §.), ita quoque omnis aequatio primi ordinis debetur alicui lineae rectae.

452. Corollarium 4.

Impossibilis est linea curva primi ordinis; quocirca curvae infimi ordinis sunt curvae secundi ordinis (451. 442. §.).

453. Problema.

Data aequatione ad curvam ZMX (5. 6. 7. 8. Fig.) *inter coordinatas* Fig. 5. 6. 7. 8.
$x = AP$, $y = MP$, *determinare angulum* TMP *inter tangentem* TM *et ordinatam* MP *in puncto contactus* M.

Solutio.

1. Si abscissa $x = AP$ augeatur incremento $\Delta x = PQ$, adeoque ordinata $MP = y$ abeat in mQ, ducaturque per puncta m, M secans LR; debebit cotangens anguli TMP aequari limiti, ad quem cotangens anguli RMP eo propius accedat, quo minus sit incrementum $\Delta x = PQ$ abscissae $x = AP$ (417. §.).

2. Ducta autem Mp parallela axi abscissarum VU; erit mp incrementum (5. 6. Fig.) vel decrementum (7. 8. Fig.) ordinatae $y = MP$ debitum incremento $\Delta x = PQ$ abscissae $x = AP$: quare debebit esse per (444. 157. §.).

$$mp = \pm\left(\frac{\delta y}{\delta x}\Delta x + \frac{\overset{2}{\delta} y}{2\,\delta x^2}\Delta x^2 + \overset{3}{\alpha}\,\frac{\overset{3}{\delta} y}{\delta x^3}\Delta x^3 + \overset{4}{\alpha}\,\frac{\overset{4}{\delta} y}{\delta x^4}\Delta x^4 + \text{etc.}\right).$$

Semper vero est (5. 6. 7. 8. Fig.) $\mathrm{Cot}\, RMP = \mathrm{Cot}\, Mmp = \frac{mp}{Mp}$: igitur semper $\mathrm{Cot}\, RMP = \pm\left(\frac{\delta y}{\delta x} + \frac{\overset{2}{\delta} y}{2\,\delta x^2}\Delta x + \overset{3}{\alpha}\,\frac{\overset{3}{\delta} y}{\delta x^3}\Delta x^2 + \overset{4}{\alpha}\,\frac{\overset{4}{\delta} y}{\delta x^4}\Delta x^3 + \text{etc.}\right)$;

eoque ipso debebit esse ob (1) per (78. §.) $\mathrm{Cot}\, TMP = \pm\frac{\delta y}{\delta x}$, adhibito signo + vel −, prout crescente abscissa $x = AP$ ordinata $y = MP$ crescit (5. 6. Fig.) vel decrescit (7. 8. Fig.).

454. Corollarium 1.

Angulus TMP in puncto contactus M inter tangentem TM et ordinatam MP ita debet determinari, ut sit $\mathrm{Cot}\, TMP = \frac{\delta y}{\delta x}$: sumetur autem δy cum signis contrariis, si crescente abscissa decrescat ordinata (453. §.).

455. Corollarium 2.

Pro situ tangentis TM et normalis MN respectu ordinatae MP et lineae abscissarum VU habebimus sequentes expressiones (454. 419. §.)

Tang

$$\text{Tang}\, NMP = \text{Tang}\, MTP = \text{Cot}\, TMP = \frac{\partial y}{\partial x}.$$

$$\text{Tang}\, MNP = \text{Tang}\, TMP = \text{Cot}\, MTP = \frac{\partial x}{\partial y}.$$

$$\text{Sin}\, MTP = \text{Cof}\, TMP = \frac{\partial y}{\sqrt{(\partial y^2 + \partial x^2)}}.$$

$$\text{Sin}\, TMP = \text{Cof}\, MTP = \frac{\partial x}{\sqrt{(\partial y^2 + \partial x^2)}}.$$

456. Corollarium 3.

Per (455. §.) vero ex (421. §.) obtinebimus pro tangente TM, fubtangente TP, normali MN, et fubnormali NP fequentes expreffiones.

$$TM = \frac{y}{\partial y}\sqrt{(\partial y^2 + \partial x^2)}, \quad TP = \frac{y\,\partial x}{\partial y}.$$

$$MN = \frac{y}{\partial x}\sqrt{(\partial y^2 + \partial x^2)}, \quad NP = \frac{y\,\partial y}{\partial x}.$$

457. Corollarium 4.

Propofita ergo aequatione ad datam curvam inter ejus coordinatas y, x, poterit tam fitus tangentis, fubtangentis, normalis, et fubnormalis refpectu ordinatae et axeos abfciffarum, quam magnitudo abfoluta per (85. 418. 420. 455. 456. §.) perfecte determinari: id tamen in fequenti capite uberius exponetur.

458. Corollarium 5.

Indeterminatum reliquimus in praecedentibus formulis (455. 456. §), utrum ordinata y, an abfciffa x inftar variabilis abfolutae fit fpectanda (4. §.): ubi vero illa ad hanc, vel haec ad illam velut certa functio ad variabilem abfolutam relata fuerit; poterit ubique poni $\partial x = 1$ cafu primo, vel $\partial y = 1$ cafu fecundo. In (453. §.) tamen pro determinatione altiorum exponentium differentialium $\overset{2}{\partial} y$, $\overset{3}{\partial} y$, $\overset{4}{\partial} y$, etc. femper debet effe $\partial x = 1$ (156. 157. §.).

459. Corollarium 6.

Fig. 5. 7. Ordinata mQ (5. 7. Fig.) occurrit tangenti TM productae in puncto q, eftque $pq = Mp.\text{Cot}\, Mqp = Mp.\text{Cot}\, TMP = \pm \frac{\partial y}{\partial x}\Delta x$, adhibito figno + vel —, prout crefcentibus abfciffis ordinatae pariter crefcunt

crescunt (5. Fig.), vel decrescunt (7. Fig.): cum igitur sub iisdem conditionibus sit $mp = \pm \frac{\varepsilon y}{\varepsilon x}\Delta x \pm \frac{\overset{2}{\varepsilon} y}{2\varepsilon x^2}\Delta x^2 \pm \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\Delta x^3 \pm$ etc. (453. §.), sitque $qm = pq - mp$ (5. Fig.), vel $qm = mp - pq$ (7. Fig.) valoris positivi pro quovis valore differentiae $\Delta x = PQ$; erit utroque casu $qm = -\frac{\overset{2}{\varepsilon} y}{2\varepsilon x^2}\Delta x^2 - \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\Delta x^3 - \overset{4}{\alpha}\frac{\overset{4}{\varepsilon} y}{\varepsilon x^4}\Delta x^4 -$ etc. semper valoris positivi, et ideo habebit exponens $\frac{\overset{2}{\varepsilon} y}{\varepsilon x^2}$ valorem negativum (77. §.).

460. Corollarium 7.

Pari ratione reperies $pq = \pm \frac{\varepsilon y}{\varepsilon x}\Delta x$, et $mp = \pm \frac{\varepsilon y}{\varepsilon x}\Delta x \pm \frac{\overset{2}{\varepsilon} y}{2\varepsilon x^2}\Delta x^2$ Fig. 6. 8.
$\pm \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\Delta x^3 \pm \overset{4}{\alpha}\frac{\overset{4}{\varepsilon} v}{\varepsilon x^4}\Delta x^4 \pm$ etc., sumto signo + vel —, prout crescentibus abscissis AP ordinatae crescunt (6. Fig.) vel decrescunt (8. Fig.): quare, cum sit $mq = mp - pq$ in (6. Fig.), et $mq = pq - mp$ in (8. Fig.) valoris positivi pro quovis valore differentiae $\Delta x = PQ$; erit utroque casu $mq = \frac{\overset{2}{\varepsilon} y}{2\varepsilon x^2}\Delta x^2 + \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\Delta x^3 + \overset{4}{\alpha}\frac{\overset{4}{\varepsilon} y}{\varepsilon x^4}\Delta x^4 +$ etc. semper valoris positivi, quod est impossibile quin etiam $\frac{\overset{2}{\varepsilon} y}{\varepsilon x^2}$ valorem positivum habeat (77. §.).

461. Corollarium 8.

Dum curva, ad quam datur aequatio inter ejus coordinatas x, y concavitatem vel convexitatem obvertit axi abscissarum, debet esse $\frac{\overset{2}{\varepsilon} y}{\varepsilon x^2}$ valoris negativi casu primo, vel positivi casu secundo, et vicissim (459. 460. §.), exponente differentiali $\overset{2}{\varepsilon} y$ ex data ad curvam aequatione pro $\varepsilon x = 1$ determinato (458. §.).

462. Corollarium 9.

Pars ordinatae mQ (5. 6. 7. 8. Fig.) inter curvam et tangentem intercepta est generatim $mq = \pm \frac{\overset{2}{\varepsilon} y}{2\varepsilon x^2}\Delta x^2 \pm \overset{3}{\alpha}\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\Delta x^3 \pm \overset{4}{\alpha}\frac{\overset{4}{\varepsilon} y}{\varepsilon x^4}\Delta x^4 \pm$ etc., Fig. 5. 6. 7. 8.

[illegible] vel convexa ex parte [illegible] puncti q a puncto con- [illegible] $=\frac{1}{\varepsilon x}\sqrt{(\varepsilon y^2+\varepsilon x^2)}$ ob pq [illegible]

[illegible]

[illegible] convergentibus [illegible] MQ, quae [illegible], quo casu [illegible]

[illegible] 2.

[illegible] $x = AQ$ est $y = z$ [illegible] $\sin\gamma$, et $\varepsilon x = z\varepsilon\gamma$ [illegible], et tang PMQ $=$ [illegible] $=\frac{[illegible]\,\sin\gamma+\varepsilon y\cos\gamma}{[illegible]\,\sin\gamma-\varepsilon x\cos\gamma}$ [illegible] expressione loco [illegible] tangente anguli [illegible] Tang TMQ, et subnormalis [illegible]

Tang TMQ $=$ [illegible] $Qr=\frac{zz}{\varepsilon\gamma}$.

[illegible] 3.

Angulus [illegible] hinc TMQ obtusus [illegible] esse angulum TMQ, [illegible] MQA. [illegible] valor $\frac{z\varepsilon y}{\varepsilon z}$ tangentis anguli TMQ, si is fiat negativus.

[illegible]

Ducatur ex puncto Q ad tangentem TMb perpendicularis $Qr = p$; erit $p = z \sin TMQ$. quare, determinato sinu anguli TMQ per ejus tangentem (§ [illegible]) fiet

$$p=\frac{z^2\varepsilon\gamma}{\sqrt{(\varepsilon z^2+z^2\varepsilon\gamma^2)}}$$

467. Co-

467. Corollarium 4.

Data aequatione ad curvam ZMX inter ordinatas $z = MQ$ sub angulis variabilibus $\gamma = MQA$ ad datum punctum Q convergentes angulosque γ, semper poterit reperiri aequatio ad eandem curvam inter ordinatam z et perpendiculum p. Cum enim sit $\frac{\partial z}{\partial \gamma} = \frac{z}{p}\sqrt{(z^2 - p^2)}$ in (466. §.), si praeterea ex data aequatione determinetur valor pro $\frac{\partial z}{\partial \gamma}$; obtinebitur ex illo et hoc valore aequatio inter z et p, excluso vel incluso angulo γ, qui hoc altero casu ex hac ipsa aequatione et data facile eliminabitur.

468. Corollarium 5.

Et vicissim, si detur aequatio ad curvam ZMX inter ordinatam $z = MQ$ et perpendiculum $p = Qr$, poterit determinari aequatio, saltem differentialis, ad eandem curvam inter z et angulum $\gamma = MQA$. Est enim $\partial\gamma = \frac{p\partial z}{z\sqrt{(z^2 - p^2)}}$: quodsi ergo ex data aequatione inter z et p determinetur valor perpendiculi p per ordinatam z, isque substituatur in $\partial\gamma$; obtinebitur aequatio inter $\partial\gamma$, ∂z et z, unde ope calculi integralis licebit quaerere aequationem inter z et γ.

469. Problema.

Data aequatione ad curvam inter coordinatas $x = AP$, $y = MP$ (5. 6. Fig.), *invenire exponentem differentialem arcus* $\mu = ZM$ *comprehensi inter ordinatam* MP *et aliam* ZC *datam in distantia* CA *ab origine abscissarum* A. Fig. 5. 6.

Solutio.

Crescente abscissa $x = AP$ incremento $PQ = \Delta x$ abeat ordinata $y = MP$ in $y' = mQ$, et arcus $\mu = ZM$ in $\mu' = Zm$: cum arcus μ et ordinata y possint spectari instar duarum functionum abscissae x, si ducatur Mp parallela axi abscissarum VU, et chorda Mm; erit $Mp = \Delta x$, $\Delta y = y' - y = mp$, et $\Delta\mu = \mu' - \mu = Mom$. Pro qualibet idcirco differentia $\Delta x = PQ$ debebit esse

$$\Delta\mu > \text{chorda } Mm, \text{ et } \Delta\mu < Mq + qm.$$

Hinc, sumta chorda $Mm = \sqrt{(\Delta y^2 + \Delta x^2)} = \frac{\Delta x}{\partial x}\sqrt{(\partial y^2 + \partial x^2)} + p\Delta x^2 + q\Delta x^3 +$ etc. pro certis coefficientibus p, q, etc. independentibus a

 Δx

Δx (84. 89. §.), ſumtisque praeterea valoribus pro Mq et mq juxta (462. §); erit

$$\Delta\mu > \frac{\Delta x}{\varepsilon x}\sqrt{(\varepsilon y^2 + \varepsilon x^2)} + p\,\Delta x^2 + q\,\Delta x^3 + \text{etc.}$$

$$\Delta\mu < \frac{\Delta x}{\varepsilon x}\sqrt{(\varepsilon y^2 + \varepsilon x^2)} \pm \frac{\overset{2}{\varepsilon} y}{2\,\varepsilon x^2}\,\Delta x^2 \pm \overset{3}{\alpha}\,\frac{\overset{3}{\varepsilon} y}{\varepsilon x^3}\,\Delta x^3 \pm \text{etc.}$$

Debet itaque eſſe per (131. §.)

$$\varepsilon\mu = \sqrt{(\varepsilon y^2 + \varepsilon x^2)}.$$

470. Corollarium.

Hinc pendet *Rectificatio* datae curvae ZMX, quae inventione arcus $\mu = ZM$ perficitur. Data nimirum aequatione ad curvam inter coordinatas y, x, quaeratur εy, ſubſtitutoque valore loco εy in expreſſione pro $\varepsilon\mu$, quaeratur arcus μ ope calculi integralis. Ceterum per ſe liquet, poni poſſe $\varepsilon x = 1$, vel $\varepsilon y = 1$, cum x vel y inſtar variabilis abſolutae poſſit ſpectari.

471. Problema.

Fig. 5. 6. *Data aequatione ad curvam* ZMX (5. 6. Fig.) *inter coordinatas* $x = AP$, $y = MP$, *invenire radium curvaturae pro puncto* M.

Solutio.

1. Ductis tangente TM et normali MN ad punctum M, cogitetur abſciſſa $x = AP$ augeri incremento $\Delta x = PQ$, tum ducatur ordinata Qm tangenti occurrens in q, et Mp parallela axi abſciſſarum VU; erit $Mp = \Delta x$, $Mq = \Delta x\sqrt{(\varepsilon y^2 + 1)} = \Delta x\,\varepsilon\mu$, et $mq = \pm \tfrac{1}{2}\overset{2}{\varepsilon} y\,\Delta x^2 \pm \overset{3}{\alpha}\overset{3}{\varepsilon} y\,\Delta x^3 \pm \overset{4}{\alpha}\overset{4}{\varepsilon} y\,\Delta x^4 \pm$ etc., conſiderata abſciſſa x inſtar variabilis abſolutae, quo fiat $\varepsilon x = 1$ (462. 469. §.).

2. Sint mk, mv perpendiculares ad tangentem TM et normalem MN; erit $Mv = mk = mq \cdot \mathrm{Sin}\,TMP$, $kq = mq\,\mathrm{Cof}\,TMP$: ob (1) per (455. 469. §.) habebimus ergo in hypotheſi variabilis abſolutae x, exponentisque $\varepsilon x = 1$.

$$Mv = \pm \frac{\overset{2}{\varepsilon} y}{2\,\varepsilon\mu}\,\Delta x^2 \pm \frac{\overset{3}{\alpha}\overset{3}{\varepsilon} y}{\varepsilon\mu}\,\Delta x^3 \pm \frac{\overset{4}{\alpha}\overset{4}{\varepsilon} y}{\varepsilon\mu}\,\Delta x^4 \pm \text{etc.}$$

$$kq = \pm \frac{\overset{2}{\varepsilon} y\,\varepsilon y}{2\,\varepsilon\mu}\,\Delta x^2 \pm \frac{\overset{3}{\alpha}\overset{3}{\varepsilon} y\,\varepsilon y}{\varepsilon\mu}\,\Delta x^3 \pm \frac{\overset{4}{\alpha}\overset{4}{\varepsilon} y\,\varepsilon y}{\varepsilon\mu}\,\Delta x^4 \pm \text{etc.}$$

3. Eſt

477. Corollarium 1.

Data aequatione ad curvam ZMX inter ejus coordinatas $x = AP$ et $y = MP$, poterunt y et εy, $\overset{2}{\varepsilon}y$ per x determinari, quo ipso poterunt etiam $u = Ap$, $z = mp$ exprimi per x: hac ergo ratione poterunt determinari duae aequationes inter u, z, x, ex quibus, eliminando variabilem x, unicam aequationem inter u et z licebit derivare (477. §.).

478. Corollarium 2.

Si cogitentur radii curvaturae ZR, M′m′, Mm, M″m″ debiti singulis punctis datae curvae ZMX; jacebunt omnia centra R, m′, m, m″ in linea curva RmS, cujus natura per naturam curvae ZMX perfecte est determinata, ita ut, data aequatione ad curvam ZMX inter ipsius coordinatas $x = AP$, $y = MP$, ex hac ipsa aequatione per (476. 477. §.) possit elici aequatio ad curvam RmS inter ejus coordinatas $u = Ap$, $z = mp$.

479. Corollarium 3.

Aequationes in (476. §.) dant per (474. 469. §.) $u = x + \frac{r\varepsilon y}{\varepsilon\mu}$, et $z = \frac{r\varepsilon x}{\varepsilon\mu} - y$: si sumas εu et multiplices per εx, εz vero ducas in εy, illudque productum ab hoc subtrahas; obtinebis $\varepsilon z\varepsilon y - \varepsilon u\varepsilon x = \frac{r}{\varepsilon\mu}(\varepsilon y\overset{2}{\varepsilon}x - \overset{2}{\varepsilon}y\varepsilon x) - (\varepsilon y^2 + \varepsilon x^2)$, unde, ob (474. 469. §.) prodit $\varepsilon z\varepsilon y - \varepsilon u\varepsilon x = 0$, et $\frac{\varepsilon u}{\varepsilon z} = \frac{\varepsilon y}{\varepsilon x}$.

480. Theorema.

Radius Mm *circuli curvedinis debiti puncto* M *datae curvae* ZMX *tangit curvam* Rms *centrorum curvedinis singulis curvae* ZMX *punctis debitorum* (478. §.) *in puncto* m, *aequaturque summae* Rm + RZ *arcus* Rm *et radii* ZR *curvedinis puncto extimo* Z *debiti.*

Demonstratio.

1. Tangat recta nm arcum Rm in m; debebit esse $\text{Tang}\, nmp = \frac{\varepsilon u}{\varepsilon z}$, et $\text{Cot}\, mNp = \text{Cot}\, MNP = \frac{\varepsilon y}{\varepsilon x}$ (455. §.): igitur erit angulus nmp complementum anguli mNp ad unum rectum (479. §.), quod est impossibile, quin tangens nm coincidat cum radio curvedinis Mm.

2. Dein-

474. Corollarium 3.

Hactenus abscissam x instar variabilis absolutae spectavimus, neglectoque exponente differentiali $\varepsilon x = 1$ et ejus quadrato $\overset{2}{\varepsilon} x^2$ loco $\frac{\varepsilon y}{\varepsilon x}$, $\frac{\overset{2}{\varepsilon} y}{\varepsilon x^2}$ retinuimus εy, $\overset{2}{\varepsilon} y$ (471. §. 1. 2. n.): quodsi ergo indeterminatum velimus relinquere, quaenam variabilium pro absoluta sumenda sit; scriptis exponentibus $\frac{\varepsilon y}{\varepsilon x}$, $\frac{\overset{2}{\varepsilon} y}{\varepsilon x^2}$ loco εy, $\overset{2}{\varepsilon} y$ in (472. §.), tum reducta expressione radii r per (410. §. 2. Schol.), erit

$$r = \frac{(\varepsilon y^2 + \varepsilon x^2)^{\frac{3}{2}}}{\varepsilon y \overset{2}{\varepsilon} x - \overset{2}{\varepsilon} y \varepsilon x} = \frac{\varepsilon \mu^3}{\varepsilon y \overset{2}{\varepsilon} x - \overset{2}{\varepsilon} y \varepsilon x} \quad (469. §.)$$

475. Corollarium 4.

Fig. 13. Quia, si abscissa x ad ordinatam y ut functio ad variabilem absolutam referatur, debet fieri $\varepsilon y = 1$, et $\overset{2}{\varepsilon} y = 0$; erit in ea hypothesi (474. §.)

$$r = \frac{(\varepsilon x^2 + 1)^{\frac{3}{2}}}{\overset{2}{\varepsilon} x} = \frac{\varepsilon \mu^3}{\overset{2}{\varepsilon} x}.$$

476. Problema.

Fig. 14. *Data aequatione ad curvam* ZMX (14. Fig.) *inter ejus coordinatas* $x = AP$, $y = MP$; *invenire distantias perpendiculares centri curvedinis debiti puncto* M *ab axibus coordinatarum* VU, DE.

Solutio.

Sit MN normalis, et in hac centrum m circuli curvedinis debiti puncto M, $mp = z$ vero et $mq = Ap = u$ sint distantiae ejus perpendiculares ab axibus VU, DE coordinatarum $x = AP$, $y = MP$; erit, ob curvedinis radium $r = mM$, $Pp = \frac{r.PN}{MN}$, $z = \frac{(r - MN)y}{MN} = \frac{ry}{MN} - y$, hinc $u = x + \frac{r.PN}{MN}$: quodsi ergo sumantur valores pro radio r, normali MN, et subnormali NP per (474. 456. §.) fiet

$$u = x + \frac{\varepsilon y(\varepsilon y^2 + \varepsilon x^2)}{\varepsilon y \overset{2}{\varepsilon} x - \overset{2}{\varepsilon} y \varepsilon x}. \quad z = \frac{\varepsilon x(\varepsilon y^2 + \varepsilon x^2)}{\varepsilon y \overset{2}{\varepsilon} x - \overset{2}{\varepsilon} y \varepsilon x} - y.$$

477. Co-

477. Corollarium 1.

Data aequatione ad curvam ZMX inter ejus coordinatas $x = AP$ et $y = MP$, poterunt y et ∂y, $\partial^2 y$ per x determinari, quo ipso poterunt etiam $u = Ap$, $z = mp$ exprimi per x: hac ergo ratione poterunt determinari duae aequationes inter u, z, x, ex quibus, eliminando variabilem x, unicam aequationem inter u et z licebit derivare (477. §.).

478. Corollarium 2.

Si cogitentur radii curvaturae ZR, M′m′, Mm, M″m″ debiti singulis punctis datae curvae ZMX; jacebunt omnia centra R, m′, m, m″ in linea curva RmS, cujus natura per naturam curvae ZMX perfecte est determinata, ita ut, data aequatione ad curvam ZMX inter ipsius coordinatas $x = AP$, $y = MP$, ex hac ipsa aequatione per (476. 477. §.) possit elici aequatio ad curvam RmS inter ejus coordinatas $u = Ap$, $z = mp$.

479. Corollarium 3.

Aequationes in (476. §.) dant per (474. 469. §.) $u = x + \frac{r\partial y}{\partial \mu}$, et $z = \frac{r\partial x}{\partial \mu} - y$: si sumas ∂u et multiplices per ∂x, ∂z vero ducas in ∂y, illudque productum ab hoc subtrahas; obtinebis $\partial z \partial y - \partial u \partial x = \frac{r}{\partial \mu} (\partial y \partial^2 x - \partial^2 y \partial x) - (\partial y^2 + \partial x^2)$, unde, ob (474. 469. §.) prodit $\partial z \partial y - \partial u \partial x = 0$, et $\frac{\partial u}{\partial z} = \frac{\partial y}{\partial x}$.

480. Theorema.

Radius Mm *circuli curvedinis debiti puncto* M *datae curvae* ZMX *tangit curvam* Rms *centrorum curvedinis singulis curvae* ZMX *punctis debitorum* (478. §.) *in puncto* m, *aequaturque summae* $Rm + RZ$ *arcus* Rm *et radii* ZR *curvedinis puncto extimo* Z *debiti.*

Demonstratio.

1. Tangat recta nm arcum Rm in m; debebit esse $\text{Tang}\, nmp = \frac{\partial u}{\partial z}$, et $\text{Cot}\, mNp = \text{Cot}\, MNP = \frac{\partial y}{\partial x}$ (455. §.): igitur erit angulus nmp complementum anguli mNp ad unum rectum (479. §.), quod est impossibile, quin tangens nm coincidat cum radio curvedinis Mm.

2. Dein-

2. Deinde, aequatione pro z in (479. §.) differentiata, tum restitutis valoribus pro $\varepsilon\mu$ et r ex (474. 469. §.), obtinebitur $\varepsilon r = \frac{\varepsilon z \, \varepsilon\mu}{\varepsilon x}$: hinc, ob $\varepsilon y = \frac{\varepsilon u \, \varepsilon x}{\varepsilon z}$ (479. §.), adeoque $\varepsilon\mu = \frac{\varepsilon x}{\varepsilon z}\sqrt{(\varepsilon u^2 + \varepsilon z^2)}$ per (469. §.) erit $\varepsilon r = \sqrt{(\varepsilon z^2 + \varepsilon u^2)}$ exponens differentialis arcus Rm (469. §.). Et ideo debet esse r = Rm + Const: cum autem pro arcu Rm = o radius curvedinis r = Mm fiat r = RZ; debet esse const = ZR, et r = Mm = Rm + ZR.

481. Theorema.

Et vicissim, si curva RmS *talem situm respectu datae curvae* ZMX *habeat, ut, ducta ad quodcunque punctum* m *tangente* Mm, *haec aequetur arcui* Rm *aucto tangente* ZR; *erit quaevis ejusmodi tangens* Mm *radius circuli curvedinis debiti puncto* M *datae curvae* ZMX, *proinde centra omnium circulorum curvedinis singulis punctis curvae* ZMX *debitorum jacebunt in curva* ZmS.

Demonstratio.

1. Cum per hypothesim Mm tangat curvam Rms in m, pro u = Ap, z = mp, μ = Rm, x = AP, y = MP; erit Tang Nmp = Cot MNP = $\frac{\varepsilon u}{\varepsilon z}$; tangens Nm = $\frac{z}{\varepsilon z}\varepsilon\mu$, et subtangens Np = $\frac{z \varepsilon u}{\varepsilon z}$ per (455. 456. 469. §.).

2. Adeoque erit $Pp = \frac{Mm.\varepsilon u}{\varepsilon\mu}$, hinc $x = \frac{u\varepsilon\mu - Mm\varepsilon u}{\varepsilon\mu}$: porro $y = \frac{Mm\varepsilon z - z\varepsilon\mu}{\varepsilon\mu}$.

3. Quodsi ergo, pro Mm = k, hinc $\varepsilon k = \varepsilon\mu$, ob Mm = Rm + ZR per hypothesin, sumas εx et εy in (2), reperies, ob $\varepsilon\mu = \sqrt{(\varepsilon z^2 + \varepsilon u^2)}$, quotientem $\frac{\varepsilon y}{\varepsilon x} = \frac{\varepsilon u}{\varepsilon z}$.

4. Est itaque Cot MNP = $\frac{\varepsilon y}{\varepsilon x}$ ob (1);(3), quod nequit subsistere quin sit MN normalis ad punctum M curvae ZMX (455. §.).

5. Radius curvedinis debitus puncto M coincidit ergo cum Mm; eidem vero radio aequari rectam Mm, sic potest ostendi. Si capias exponentem differentialem εy in (2), obtinebis ob (3) $k = Mm = \frac{\varepsilon y \varepsilon\mu^2}{\overset{2}{\varepsilon}\mu\varepsilon z - \overset{2}{\varepsilon} z\varepsilon\mu}$.

Est

Est autem per (3) $\frac{\partial z}{\partial \mu} = \frac{\partial x}{\sqrt{(\partial y^2 + \partial x^2)}}$, unde differentiando producitur $\partial\mu\,\partial^2 z - \partial z\,\partial^2\mu = \frac{(\partial y\,\partial^2 x - \partial x\,\partial^2 y)\,\partial y\,\partial\mu^2}{(\partial y^2 + \partial x^2)^{\frac{3}{2}}}$: hoc igitur valore substituto in praecedenti expressione pro k obtinebitur k = radio curvedinis in puncto M (474. §.).

482. Definitio.

Si linea abscissarum VU (15. Fig.) tangat datam curvam RS in R, sumaturque pars ejus constans ZR, tum cogitetur ad quodvis punctum m ejusdem curvae duci tangens Mm = Rm + RZ, nimirum aequalis summae arcus Rm et rectae constantis ZR; jacebunt puncta extima M omnium tangentium in certa curva ZMX, cujus *Evolutam* vocant curvam RmS. Si, inquiunt, concipiatur curvae RmS filum ZRmS ita adplicari, ut ejus pars ZR eandem in R tangat; tum cogitetur extremum Z aequabiliter versus X progredi, ut semper pars fili Mm curvam deserens in rectam protensa sit, quae curvam in m tangat; curva RS evolvatur in lineam rectam determinatam per ejus tangentem XS = RS + RZ, punctumque Z generabit curvam ZMX: ob hanc ipsam causam vocant curvam RmS *evolutam*, et ZMX curvam *evolutione prioris genitam*. Fig. 15.

483. Corollarium 1.

In evoluta RmS curvae ZMX jacent centra omnium circulorum curvedinis singulis punctis curvae ZMX debitorum, ita ut, si ad quodcunque punctum m evolutae ducatur tangens Mm curvae ZMX in M occurrens, punctum m debeat esse centrum, et Mm radius circuli curvaturae debiti puncto M (481. §.).

484. Corollarium 2.

Data aequatione ad curvam ZMX evolutione alterius curvae RmS genitam inter coordinatas x = AP, y = MP, poterit, ob (483. §.), per (476. 477. 478. §.) determinari aequatio ad evolutam RmS inter coordinatas u = Ap, z = mp.

485. Corollarium 3.

Cum sit tangens Mm = Rm + ZR radius curvedinis in puncto M curvae ZMX (483. §.); erit quilibet arcus Rm evolutae RmS algebraice rectificabilis, si radius curvedinis in quovis puncto M curvae evolutione prioris genitae aequetur functioni algebraicae.

486. Pro-

486. Problema.

Data aequatione ad evolutam RmS *inter ejus coordinatas* $Rq = v$, $mq = t$, *invenire aequationem ad curvam* ZMX *evolutione prioris genitam inter ipsius coordinatas* $ZP = x$, $MP = y$.

Solutio.

1. Nimirum linea abscissarum RQ datae evolutae RmS est perpendicularis ad ejus tangentem VU in R, quae lineam abscissarum pro curva ZMX evolutione prioris genita exhibet. Cum jam sit tangens $Mm = mR + ZR = r$ radius curvedinis in puncto M, posito arcu $Rm = s$, et $ZR = a$; erit $r = s + a$. Porro est $v = Rq = mp$, et $t = mq = Rp = Ap - a$: igitur per (479. §.) reperietur $y = \frac{(s+a)\,\partial x}{\partial \mu} - v$, et $x = t + a - \frac{(s+a)\,\partial y}{\partial \mu}$.

2. Deinde $t = Ap - a = u - a$ dat $\partial t = \partial u$, et $v = Rq = mp = z$ dat $\partial v = \partial z$: quamobrem, si pro his valoribus per (480. §. 2. n.) determines $\frac{\partial y}{\partial \mu}$ et $\frac{\partial x}{\partial \mu}$, obtinebis ex (1)

$$y = \frac{(s+a)\,\partial v}{\sqrt{(\partial t^2 + \partial v^2)}} - v, \text{ et } x = t + a - \frac{(s+a)\,\partial t}{\sqrt{(\partial t^2 + \partial v^2)}}.$$

3. Proposita ergo aquatione ad curvam RmS inter coordinatas $v = Rq$, $t = mq$, determinetur ∂t per ∂v: hoc enim valore loco ∂t in (2) substituto obtinebuntur expressiones pro y, x per v, t, s: ex his demum aequationibus et aequatione inter v, t data licebit eliminare v, t, eoque ipso unam aequationem inter x et y inde derivare, quae tamen ab arcu $s = Rm$ haud liberabitur.

487. Corollarium.

Natura curvae ZMX pendet a rectificatione illius evolutae RmS: si arcus $s = Rm$ fuerit functio algebraica ordinatae $mq = t$; erit etiam aequatio inter x, y ad curvam ZMX algebraica, transcendens vero, si arcus $s = Rm$ aequatur functioni transcendenti ordinatae $mq = t$ (486. §.).

488. Problema.

Fig. 5. *Data aequatione ad curvam* ZMX (5. Fig.) *inter coordinatas* $x = AP$, $y = MP$; *invenire spatium* MPCZ *inter ordinatam* MP *et aliam* ZC *datam in distantia* CA *ab origine* A *abscissarum.*

Solutio.

Sit spatium $MPCZ = S$, quod instar unius functionis abscissae $x = AP$ potest spectari. Si abscissa $x = AP$ capiat incrementum $\Delta x = PQ$, ducaturque ordinata Qm, et Mp parallela axi abscissarum VU, altera vero parallela ml ordinatae MP occurrat in l; erit spatium $PMmQ = \Delta S$, $mp = \Delta y$, $Mp = \Delta x$; hinc $\Delta S > MpQP$ et simul $\Delta S < lmQP$ pro qualibet differentia Δx. Habebimus igitur $\Delta S > y\Delta x$ et simul $\Delta S < (y + \Delta y)\Delta x$, unde ob (84. §.) per (131. §.) obtinetur exponens differentialis $\partial S = y\partial x$, cujus integrale aequabitur spatio S.

489. Problema.

Data aequatione ad curvam AX (16. 17. Fig.) *inter ordinatas* $z = BC$ Fig. 16. 17. *ad datum punctum* C *convergentes et angulos variabiles* $ACB = \gamma$, *quadrare sectorem* ACB, *seu invenire ejus aream* S.

Solutio.

Angulus $\gamma = ACB$ capiat incrementum $BCD = \Delta\gamma$, et ex puncto C radiis $CB = z$, CD describantur arcus circulares Bd, Db; erit $Dd = Bb = \Delta z$ differentia ordinatae $BC = z$, et area sectoris $BCD = \Delta S$ differentia areae $S = BCA$ (17. 18. §.): si ergo ordinata z spectetur instar certae functionis anguli γ; poterit poni $\Delta z = \alpha\Delta\gamma + \beta\Delta\gamma^2 + \text{etc.}$ (84. §.).

Iam vero patet, aream ΔS sectoris BDC pro quovis angulo $\Delta\gamma$ ita se habere ad areas sectorum circularium BdC, bDC, ut debeat esse $\Delta S > BdC$ et simul $\Delta S < bDC$ (16. Fig.), vel $\Delta S > bDC$ et $\Delta S < BdC$ (17. Fig.): ea propter, cum sit arcus circularis $Bd = \Delta\gamma . BC = \Delta\gamma . z$, et $bD = \Delta\gamma . DC$, seu $bD = \Delta\gamma(z + \Delta z)$ in (16. Fig.), et $bD = \Delta\gamma(z - \Delta z)$ in (17. Fig.) per (5. §. 3. Schol.); debebit esse $\Delta S > \frac{1}{2}z^2\Delta\gamma$, et $\Delta S < \frac{1}{2}z^2\Delta\gamma + (z + \frac{1}{2}(\alpha\Delta\gamma + \beta\Delta\gamma^2 + \text{etc.}))(\alpha\Delta\gamma^2 + \beta\Delta\gamma^3 + \text{etc.})$; vel $\Delta S < \frac{1}{2}z^2\Delta\gamma$, et $\Delta S > \frac{1}{2}z^2\Delta\gamma + (\frac{1}{2}(\alpha\Delta\gamma + \beta\Delta\gamma^2 + \text{etc.}) - z)(\alpha\Delta\gamma^2 + \beta\Delta\gamma^3 + \text{etc.})$.

Est igitur $\partial S = \frac{1}{2}z^2\partial\gamma$ (131. §.) exponens differentialis areae S, cujus integratio dabit idcirco ipsam aream $S = ABC$.

490. Problema.

Data aequatione ad curvam BPC (18. Fig.) *inter ejus coordinatas* Fig. 18. $x = Ba$, $y = aP$, *cujus revolutione circa axem abscissarum* AE *cogitetur generari corpus rotundum* BCD; *invenire soliditatem ejusdem corporis.*

Solutio.

Sectio PmpnP in a perpendicularis ad axem AE abscindit segmentum PBpnPm$=$S, quod instar unius functionis abscissae x$=$Ba potest spectari. Si itaque abscissa x$=$Ba augeatur incremento $\Delta x$$=$ab, cogiteturque per punctum b duci sectio rescr priori parallela; exprimet $\Delta S$$=$Ppsr partem totius corporis comprehensam inter sectiones PmpnP et rescr, quae idcirco pro quavis differentia $\Delta x$$=$ab erit major cylindro basees PmpnP et altitudinis ab, simul autem minor cylindro baseos rescr et ejusdem altitudinis ab: igitur erit pro quavis differentia $\Delta x$$=$ ab.

$$\Delta S > PmpnP.\Delta x \text{ et } \Delta S < rescr.\Delta x.$$

Habet autem sectio circularis PmpnP ordinatam aP$=$y, et sectio rescr ordinatam rb$=$sb$=$rR$+$Pa$=$sS$+$pa$=\Delta y+y$ pro radio: pro ratione $1:\pi$ radii ad semiperipheriam debet igitur esse PmpnP$=\pi y^2$, et rescr$=\pi(y+\Delta y)^2$. Adeoque habebimus etiam

$$\Delta S > \pi y^2 \Delta x \text{ et } \Delta S < \pi y^2 \Delta x + 2\pi y \Delta y \Delta x + \pi \Delta y^2 \Delta x.$$

Quare per (131. §.) debet esse exponens differentialis $\partial S = \pi y^2 \partial x$, cujus integratio dat soliditatem segmenti S$=$BPnpmP.

491. Problema.

In eadem hypothesi (490. §.) *invenire superficiem rotundam segmenti* PBpnPmp.

Solutio.

1. Recta AF tangat curvam BC in P, et abscissae Bb$=$Ba$+$ab$=x+\Delta x$ respondeat ordinata rb$=$rR$+$Rb$=$rR$+$Pa$=\Delta y+y$, quae producta occurrat tangenti AF in Q: posito igitur arcu BP$=\mu$ ponatur arcus Pr$=\Delta\mu$.

2. Quare, dum curva BC sua revolutione circa axem AE corpus rotundum CBD generat, concipi potest arcus BP generare superficiem rotundam segmenti PBp, quae dicatur S, arcus vero Br superficiem rotundam segmenti rBs, ita ut differentia ΔS prioris superficiei aequari debeat superficiei rotundae partis hujus segmenti interceptae inter sectiones PmpnP, rescr in punctis a, b ad axem AE perpendiculares.

3. In eadem hypothesi (2) generant pars PQ tangentis AF, et chorda Por conos truncatos PpqQP, PposroP; pars vero Qr ordinatae br productae in Q intercepta inter tangentem et curvam generat annulum diorum rb, Qb.

4. Est autem pro quavis differentia $\Delta x = ab$ abscissae $x = Ba$ superficies rotunda ΔS in (2) major superficie rotunda coni truncati Porsop, et simul minor summa superficiei rotundae coni truncati PQqp, et superficiei planae annuli radiorum rb, Qb in (3).

5. Superficies rotunda coni truncati Porsop, sumta ratione $1:\pi$ radii ad semiperipheriam, est $\pi(2y+\Delta y)Por = \pi(2y+\Delta y)\sqrt{(\Delta y^2+\Delta x^2)}$.

6. Superficies vero rotunda coni truncati PQqp est $\pi(2y+\Delta y+rQ)PQ$; et superficies plana annuli radiorum rb, Qb est $\pi(2y+2\Delta y+Qr)Qr$.

7. Quamobrem, si $\sqrt{(\Delta y^2+\Delta x^2)}$ exprimatur per (469. §.), et rQ, PQ in (5) (6) determinentur ut mq, Mq in (462. §.), obtinebimus ex (5) (6) (4) pro certis coefficientibus k, l, m, - - - K, L, M, etc. sequentes expressiones.

$$\Delta S > \frac{2\pi y}{\partial x}\Delta x\sqrt{(\partial y^2+\partial x^2)}+k\Delta x^2+l\Delta x^3+m\Delta x^4+\text{etc.}$$

$$\Delta S < \frac{2\pi y}{\partial x}\Delta x\sqrt{(\partial y^2+\partial x^2)}+K\Delta x^2+L\Delta x^3+M\Delta x^4+\text{etc.}$$

Per (131. §.) est itaque $\partial S = 2\pi y\sqrt{(\partial y^2+\partial x^2)} = 2\pi y\,\partial\mu$ (469. §.) Exponens differentialis functionis S aequalis superficiei rotundae segmenti PBp, cujus integratio dabit superficiem S.

CAPUT

CAPUT IX.
DE
QUIBUSDAM CURVIS SPECIATIM.

492. Problema.

Admissa resolutione aequationum omnis ordinis, explorare situm curvae, ad quam data sit aequatio inter ejus coordinatas x, y, *respectu axium harum coordinatarum.*

Solutio.

1. Data ad curvam aequatio sit $\alpha + \overset{x}{\varphi} + \overset{y}{\mu} + \overset{xy}{\psi} = o$ (443. §.), quae ita sit comparata, ut pro $x = o$ vel $y = o$ omnes termini complectentes x casu primo, vel y casu secundo, fiant aequales nihilo.

2. Quaelibet ordinata y exhibet distantiam perpendicularem unius puncti curvae ab axe abscissarum; et quaevis abscissa x dat distantiam ejusdem puncti curvae ab axe ordinatarum aequalem distantiae puncti axeos abscissarum, in quo ei insistit ordinata y, ab origine abscissarum (414. 411. 447. §.).

3. Fiat igitur in data aequatione ad curvam ordinata $y = o$; abibit ea in aequationem determinatam $\alpha + \overset{x}{\varphi} = o$ ob (1), per quam determinantur abscissae x, quibus respondent ordinatae $y = o$, in quarum idcirco punctis extimis curva axi abscissarum occurrit, eundemque secat vel tangit (2). Quaevis ergo radix realis aequationis $\alpha + \overset{x}{\varphi} = o$, positiva vel negativa, dabit unam abscissam x, positivam vel negativam (413. §.), aequalem distantiae unius puncti axeos abscissarum ab harum origine, in quo curva occurrit eidem axi: toties itaque occurret curva axi abscissarum, quot radices reales habuerit aequatio $\alpha + \overset{x}{\varphi} = o$: si quae harum radicum inter se aequales fuerint, omnes positivae, vel omnes negativae, indicium erit, totidem curvae ramos in uno eodemque puncto occurrere axi abscissarum: si autem omnes radices aequationis $\alpha + \overset{x}{\varphi} = o$ fuerint imaginariae; signum erit, nullum extare punctum in axe abscissarum, in quo ei curva occurrat.

4. Sic quoque pro x=o abibit data aequatio (1) in aequationem determinatam $\overset{y}{\alpha} + \overset{y}{\mu} = o$, quae determinat ordinatas respondentes abscissis x=o, ordinatas idcirco in origine abscissarum coincidentes cum axe ordinatarum, aequales distantiis totidem punctorum hujus axeos ab origine abscissarum, in quibus ei curva occurrit (2). Quamobrem occurret curva axi ordinatarum in tot punctis, quot radices reales inaequales habuerit aequatio $\alpha + \overset{y}{\mu} = o$: si quae vero ejus radices reales fuerint aequales, omnes positivae, aut omnes negativae; determinabunt hae unum punctum in axe ordinatarum, in quo ei totidem rami curvae occurrent: si autem omnes radices illius aequationis fuerint imaginariae; indicabunt eae, nullum dari punctum in axe ordinatarum, in quo ei curva possit occurrere.

5. Denique sumatur quaecunque determinata abscissa x=±a positiva vel negativa, abibit data ad curvam aequatio (1) in aequationem determinatam $\alpha + \overset{\pm a}{\varphi} + \overset{\pm ay}{\mu} = o$. Quaelibet radix realis hujus aequationis, positiva vel negativa, dabit unam curvae ordinatam y, pariter positivam vel negativam, respondentem abscissae x=±a, quo ipso determinabitur unum punctum curvae in distantia x=a ab axe ordinatarum et distantia y ab axe abcissarum (2). Si quae autem reales radices fuerint inter se aequales, omnes positivae vel omnes negativae; omnes unum tantum punctum curvae in distantia =a ab axe ordinatarum et distantia =y ab axe abscissarum determinabunt, in quo totidem curvae rami concurrent: radices demum omnes imaginariae denotabunt, nullum curvae punctum jacere in distantia x=±a ab axe ordinatarum.

493. Corollarium 1.

Cum aequatio ad curvam inter coordinatas x, y pro y=o nequeat esse altioris ordinis, quam est ipsa curva (442. §.), nec ulla aequatio determinata plures radices possit habere, quam unitates exponens ordinis complectitur, ad quem ea pertinet (175. §.); perspicuum est, nullam curvam posse axi abscissarum in pluribus punctis occurrere, quam unitates continentur in exponente ordinis, ad quem curva spectat; fieri tamen posse, ut aliqua curva ei in paucioribus punctis, alia vero in nullo puncto occurrat (492. §. 3. n).

494. Corollarium 2.

Omnis recta, quemcunque illa situm respectu datae curvae habeat, potest sumi pro axe abscissarum (412. §.): impossibile ergo est, ut aliqua recta datae

datae cuicunque curvae in pluribus punctis occurrat, quam unitates habet exponens ordinis, ad quem curva pertinet (449. 493. §.).

495. Corollarium 3.

Recta, quae per punctum curvae *2plex*, *3plex*, aut generatim n*tuplex* transit, censenda est curvae eo ipso in 2, 3, aut generatim n punctis occurrere (440. §.): quare, cum possibile sit, rectam per punctum n*tuplex* curvae ita ducere, ut illa adhuc per aliquod aliud ejusdem curvae punctum transeat, curvae idcirco ad minimum in punctis numero n+1 occurrat, impossibile est, ut aliqua curva aut n*tuplex* punctum habeat (440. §.), cujus index n exponentem ordinis, ad quem curva pertinet (442. §.), exaequet aut superet, aut bina habeat puncta multiplicia, e. gr. punctum m*tuplex* et aliud r*tuplex*, quorum indices m+r simul exponentem ordinis curvae superent (494. §.).

496. Corollarium 4.

Curvae secundi ordinis non possunt habere puncta multiplicia, consequenter nec cuspides, neque nodos (442. 441. 495. §.).

497. Corollarium 5.

Curva, ad quam datur aequatio inter coordinatas x, y habebit ramum juxta axem abscissarum in indefinitum excurrentem, si abscissis x ultra certum valorem = a indesinenter crescentibus semper aliquae ordinatae y, omnes positivae vel omnes negativae, respondeant: et si hae continuo fuerint minores; erit linea abscissarum asymptotus ejus rami (492. §. 5. n. et 437. §.).

498. Corollarium 6.

Si data ad curvam aequatio sic sit comparata, ut determinatae alicui abscissae x = k ordinatae y reales, una vel plures, abscissis vero x = k ± e quantitate e quomodocunque parva auctis aut minutis ordinatae tantum imaginariae respondeant; habebit curva unum vel plura puncta in distantia k ab axe ordinatarum ita sejuncta a reliquis ejusdem curvae partibus, ut nullus arcus curvae in iis punctis initium possit sumere, aut terminari (492. §. 5. n.): puncta ejusmodi *conjugata* vocantur.

499. Corollarium 7.

Tangens ad punctum conjugatum curvae est imaginaria; et reale curvae punctum, cui non nisi imaginaria tangens respondet, debet esse conjugatum (498. 415. §.).

500. Pro-

500. Problema.

Data ad curvam aequatione $Z=0$ *inter coordinatas* x, y, *determinare situm tangentis et subtangentis pro dato puncto.*

Solutio.

1. Sit UV (19. 20. Fig.) axis abscissarum, et illarum origo punctum Fig. 19.20.
A vel B. Punctum curvae, ad quod tangens ducta cogitetur, sit M, determinatum per ordinatam positivam vel negativam PM, et abscissam pariter positivam aut negativam AP, vel BP (413. §.).

2. Cum data aequatio $Z=0$ complectatur functionem Z duarum variabilium y, x, erit ${}^{y}\partial Z + {}^{x}\partial Z = 0$ (141. §.): igitur, si sit ${}^{y}\partial Z = A\partial y$ et ${}^{x}\partial Z = B\partial x$, habebimus $\frac{\partial y}{\partial x} = -\frac{B}{A}$.

3. Iam vero, per hypothesin, datur punctum contactus M, cis vel trans axem abscissarum UV (1), unde per se elucet, tangentem TM casu primo cis, et casu secundo trans axem abscissarum UV cadere. Attendatur porro ad exponentem differentialem ∂y ex data aequatione $Z=0$ derivatum (2): si enim is pro data abscissa $x=AP$, vel $x=BP$, et ordinata $y=MP$, utraque absolute spectata, positivum valorem habeat, vel negativum, constabit eo ipso, crescente abscissa $x=AP$ vel $x=BP$ ordinatam $y=MP$ crescere casu primo, et decrescere casu secundo (444. §.); consequenter jacebit tangens TM cum subtangente TP ad eandem partem ordinatae MP, ad quam jacet origo abscissarum A vel B, casu primo, ad partem vero oppositam casu secundo (418. §.).

4. Quamobrem, quo situs tangentis et subtangentis respectu axeos abscissarum UV ordinataeque MP perfecte definiatur, determinari debet adhuc magnitudo absoluta angulorum MTP, TMP: hi autem determinabuntur per (455. §.), si, spectata magnitudine absoluta abscissae x et ordinatae y, exponens differentialis ∂y (2) pro illis abscissis, pro quibus crescentibus decrescunt ordinatae (3), cum signis contrariis sumatur (454. §.), ita ut fiat

$$\text{Tang } MTP = \text{Cot } TMP = \frac{\pm \partial y}{\partial x};$$

$$\text{Tang } TMP = \text{Cot } MTP = \frac{\partial x}{\pm \partial y};$$

signo — pro eo casu adhibito, quo ∂y fuerit valoris negativi (3).

 Scho-

Scholion.

Fig. 1. Exemplo sit circulus (1. Fig.) radio $r=AC$ descriptus. Si diameter AB sit axis, et A origo abscissarum; erit $y^2=2rx-x^2$ aequatio ad circulum, eaque dabit $\varepsilon y=\frac{(r-x)\varepsilon x}{y}$. Expressio haec retinebit sua signa, si etiam $-y$ loco y sumatur: cum enim εy ex $y=\pm\sqrt{(2rx-x^2)}$ nascatur; debebit utique, sumta ordinata $-y$, sumi etiam $-\varepsilon y$. Quare, cum hoc casu abscissae negativae locum non habeant, crescente abscissa positiva $x=AQ$ usque $x=AC=r$ habebit εy valorem positivum, eodem vero crescente ultra $x=AC=r$ obtinebit εy valorem negativum: hinc ergo per (3) colligimus, tangentem UP et Uq cum subtangente UQ pro quovis puncto P, q primi quadrantis AD et quarti AE ad eandem cum origine abscissarum A partem ordinatae cadere; tangentem vero Vp et VM cum subtangente Vm pro quolibet puncto p, M secundi quadrantis DB et tertii BE ad partem ordinatae ei oppositam cadere, ad quam jacet origo abscissarum A. Quodsi demum m denotet angulum ad U vel V, sub quo tangens secat axem AB, et n designet angulum ad P, q, M, vel p, sub quo tangens occurrit ordinatae, habebis per (4) sequentes expressiones, signo $-$ pro secundo et tertio quadrante adhibito, ob $x=Am>r$.

$$\text{Tang}\, m=\pm\frac{r-x}{y}. \qquad \text{Tang}\, n=\pm\frac{y}{r-x}$$

Pro abscissis u a centro C computatis esset aequatio ad circulum $y^2=r^2-u^2$, hinc $\varepsilon y=\frac{-u\varepsilon u}{y}$: erit igitur εy pro omni abscissa et ordinata valoris negativi, unde sequitur per (3), tangentem cum subtangente pro quovis circuli puncto cadere ad partem ordinatae ei oppositam, ad quam jacet centrum C sumtum pro origine abscissarum: per (4) autem, retenta praecedente literarum m, n significatione, habebimus $\text{Tang}\, m=\frac{u}{y}$ et $\text{Tang}\, n=\frac{y}{u}$. Plura exempla in sequentibus se sponte offerent.

501. Corollarium 1.

Cognito situ tangentis et subtangentis (500. §.), innotescet eo ipso situs normalis et subnormalis (420. §.). Normalis nimirum cum subnormali semper cadit ad partem ordinatae illi oppositam, ad quam cadit tangens cum subtangente: quodsi ergo immediate ex data ad curvam aequatione

Z=o

$Z = 0$ elicias differentiando exponentem differentialem εy (500. §. 2. n.), habebit is pro datis coordinatis, absolute consideratis, valorem positivum vel negativum, qui indicabit, normalem cum subnormali casu secundo ad eandem partem ordinatae, ad quam jacet origo abscissarum, et casu primo ad partem illi oppositam cadere (500. §. 3. n.).

502. Corollarium 2.

Dato situ tangentis, subtangentis, normalis, et subnormalis (500. 501. §.), determinabitur per (456. §.) magnitudo absoluta harum linearum, modo, spectata magnitudine absoluta coordinatarum, εy pro illis abscissis, pro quibus crescentibus decrescunt ordinatae, pro quibus idcirco εy valorem negativum induit (444. §.), cum signis contrariis sumatur (454. §.).

Scholion.

Sic e. gr. in (500. §. Schol.), si abscissae a vertice diametri computentur, erit in circulo tangens $\pm \frac{ry}{r-x}$, subtangens $\pm \frac{y^2}{r-x}$; ubi signum $-$ adhiberi debet eo determinato casu, quo fit $x > r$: si autem abscissae computentur a centro, debebit esse tangens $\frac{ry}{u}$, et subtangens $\frac{y^2}{u}$.

503. Corollarium 3.

Si aequatio ad curvam $Z = 0$ det ${}^{y}\varepsilon Z + {}^{x}\varepsilon Z = A\varepsilon y + B\varepsilon x = 0$ Fig. 19. 20. (500. §. 2. n.), sitque $A = 0$ vel $B = 0$, adeoque $\frac{\varepsilon x}{\varepsilon y} = 0$ vel $\frac{\varepsilon y}{\varepsilon x} = 0$; erit casu primo Tang TMP (19. 20. Fig.), et casu altero Tang MTP aequalis nihilo: primo igitur casu erit tangens perpendicularis ad axem abscissarum UV, et casu altero perpendicularis ad ordinatam, proinde parallela axi UV.

504. Corollarium 4.

Verum fieri potest, ut utraque pars ${}^{y}\varepsilon Z$ et ${}^{x}\varepsilon Z$ aequationis differentialis $\varepsilon Z = 0$ aequetur nihilo, quo casu fiet $\frac{\varepsilon y}{\varepsilon x} = \frac{0}{0}$ (503. §.): hoc casu aequatio $\varepsilon Z = 0$ ad determinandum situm tangentis haud sufficiet. Sumatur idcirco secunda aequatio differentialis $\overset{2}{\varepsilon} Z = 0$, adeoque ${}^{y}\overset{2}{\varepsilon} Z + 2\,{}^{y}\varepsilon.{}^{x}\varepsilon Z + {}^{x}\overset{2}{\varepsilon} Z = 0$ (152. §.), seu $A\varepsilon y^2 + 2B\varepsilon y\varepsilon x + C\varepsilon x^2 = 0$, si fit ${}^{y}\overset{2}{\varepsilon} Z = A\varepsilon y^2$, ${}^{y}\varepsilon.{}^{x}\varepsilon Z = B\varepsilon y\varepsilon x$, et ${}^{x}\overset{2}{\varepsilon} Z = C\varepsilon x^2$: hinc enim nascetur aequatio quadratica

$\frac{A}{C}+\frac{2B}{C}\frac{\partial x}{\partial y}+\frac{\partial x^2}{\partial y^2}=0$, cujus binae radices exhibebunt duos valores pro $\frac{\partial x}{\partial y}=\text{Tang}\,TMP$ (500. §. 4. n.). Si utraque radix fuerit realis, binae tangentes in M pertinebunt ad binos curvae ramos in M concurrentes, punctumque M erit duplex (440. §.): et si hae radices inter se aequales sint, unica tangens in M ad duos ramos sese in M tangentes pertinebit: quodsi autem ambae radices fuerint imaginariae; erit M punctum conjugatum curvae (499. §.).

505. Corollarium 5.

Et generatim, si ex data ad curvam aequatione $Z=0$ per (152. 153. §.) determinentur successive aequationes differentiales $\partial Z=0$, $\overset{2}{\partial} Z=0$, $\overset{3}{\partial} Z=0$, - - - $\overset{r-1}{\partial} Z=0$, singulae vero partes singularum aequationum pro dato curvae puncto deprehendantur aequari nihilo; sumatur aequatio differentialis ordinis proxime altioris $\overset{r}{\partial} Z=0$, unde, singulis partibus exponentis $\overset{r}{\partial} Z$ per (153. §.) determinatis, tum per potentiam ∂y^r et coefficientem potentiae ∂x^r divisis, nascetur aequatio *rti* ordinis formae $\alpha+\beta\frac{\partial x}{\partial y}+\gamma\frac{\partial x^2}{\partial y^2}+ - - - - \frac{\partial x^r}{\partial y^r}+=0$. Quodsi jam omnes radices hujus aequationis fuerint reales, exhibebunt eae totidem valores reales pro tangente $\frac{\partial x}{\partial y}$ anguli TMP, quo casu idcirco numero r tangentes ad totidem curvae ramos in puncto M concurrentes pertinebunt, punctumque M erit *rtuplex* (440. §.): si autem omnes radices fuerint imaginariae; erit M punctum conjugatum (498. §.): in genere demum pro n radicibus realibus ejusdem aequationis erit M punctum *ntuplex* (440. §.).

506. Corollarium 6.

Hinc patet, qua ratione liceat explorare, an curva, ad quam datur aequatio $Z=0$ inter indeterminatas coordinatas x, y, puncta multiplicia habeat, quave ratione coordinatae x, y, quibus ea puncta respondeant, possint determinari. Sumta nimirum prima aequatione differentiali $\partial Z = {}^y\partial Z + {}^x\partial Z = 0$ (152. §.), fiat ${}^y\partial Z=0$, ${}^x\partial Z=0$, sicut est $Z=0$, tum ex una trium harum aequationum exprimatur valor alterutrius coordinatae x, y, isque substituatur in reliquis duabus aequationibus: quoties enim hae aequationes communes aliquas radices habuerint; toties poterunt

per

per has radices determinari coordinatae x, y, quibus respondent puncta multiplicia: utrum vero haec puncta sint duplicia, vel triplicia etc. per (504. 505. §.) licebit investigare.

507. Corollarium 7.

Ubicunque adsunt cuspides seu puncta reflexionis, aut puncta flexus contrarii, vel nodi, adsunt eo ipso puncta multiplicia (441. §.): quodsi ergo certo constet, curvam, ad quam datur aequatio, nulla habere puncta multiplicia (504. 505. 506. §.), constabit eo ipso, eandem curvam nec cuspides habere, neque puncta flexus contrarii, vel nodos. Ex eo tamen, quod aliqua curva puncta multiplicia habeat, haud sequitur, eandem quoque cuspides vel puncta flexus contrarii, aut nodos habere (440. 441. §.).

Scholion.

Peculiariter idcirco debet investigari, utrum datum curvae punctum multiplex sit cuspis, punctum flexus contrarii, vel nodus. Verum has aliasque complures disquisitiones generales, quae, nisi uberius, quam fieri solet, exponantur, tyronem facile in errorem inducunt, intactas relinquemus, ne ab instituti ratione nimiopere abducamur: iis itaque substituemus adplicationem praecedentis theoriae ad peculiares quaspiam curvas, algebraicas et transcendentes, quarum aliquae in sequentibus erunt usui.

508. Definitio.

Inter curvas algebraicas maxime memorabiles sunt *Sectiones conicae*. Curva, cujus naturam exprimit aequatio $y^2 = px$ inter coordinatas x, y rectamque constantem p, vocatur *Parabola*, et p ejus *Parameter*.

509. Corollarium 1.

Sit VU (21. Fig.) axis, et A origo abscissarum, DE autem axis ordinatarum; respondebit in parabola cuivis abscissae positivae $x = AP$ duplex ordinata $y = \pm\sqrt{p.AP}$ (508. §.), una positiva, puta $y = MP$, et altera negativa $y = Pm$, priori aequalis, utraque eo major, quo major est abscissa $x = AP$; abscissis autem negativis $x = -Ae$ non aliae quam imaginariae ordinatae $y = \sqrt{-p.Ae}$ respondebunt (508. §.): et pro abscissa $x = AP = 0$ fiet etiam ordinata $y = MP = 0$ (508. §.). Fig. 21.

510. Corollarium 2.

Parabola constat ergo duobus ramis AR, AS in origine abscissarum A concurrentibus, qui juxta axem abscissarum VU in indefinitum excur-

 runt,

runt, eo magis ab eo recedentes, quo magis versus U producuntur. Linea vero abscissarum VU bisfecabit omnem chordam Mm ei perpendicularem, et ideo dividet illa spatium MAm in duas partes AMP, AmP congruentes, proinde inter se similes et aequales (509. §.). Eapropter appellatur A *Vertex* parabolae, AU autem est ejus *Diameter absoluta*, et in specie *Axis* (438. §.).

511. Corollarium 3.

Si ex vertice A ad quodcunque parabolae punctum M ducatur recta AM; erit $AM^2 = MP^2 + AP^2$: igitur $AM^2 = (p+x)x$ (508. §.)

512. Corollarium 4.

Tangens TM (vel Tm) ad quodvis parabolae punctum M (vel m) cum subtangente TP debet jacere ad eandem partem ordinatae MP (vel mP), ad quam jacet ejus vertex A; normalis vero MN (vel mN) cum subnormali NP cadet ad partem oppositam (508. 500. §. 3. n.) 501. §.): posita abscissa $x = AP$, et ordinata $y = MP$, obtinebuntur per (508. 456. 455. §.) sequentes expressiones pro tangente, subtangente, normali et subnormali, situque harum rectarum respectu axeos et ordinatae.

$$TM = \sqrt{(px + 4x^2)};\ TP = 2x;\ MN = \sqrt{(\tfrac{1}{4}p^2 + px)};$$

$$NP = \tfrac{1}{2}p;\ \mathrm{Sin}\,MTP = \mathrm{Cof}\,TMP = \sqrt{\left(\frac{p}{p+4x}\right)}.$$

513. Corollarium 5.

Subnormalis parabolae est constans, nimirum aequalis semiparametro; subtangens vero aequatur duplae abscissae; et tangens in vertice A fit perpendicularis ad axem AU (512. §.).

514. Corollarium 6.

Pro arcu $\mu = AM$, cum sit $\partial x = \frac{2y\partial y}{p}$ (508. §.); erit per (469. §.) $\partial\mu = \frac{\partial y}{p}\sqrt{(p^2 + 4y^2)}$, unde, integrando per (289. §.) reperietur arcus $\mu = Z + C$ pro quavis abscissa $x = AP$, qui pro $x = 0$, debebit pariter aequari nihilo, quo ipso valor constantis C poterit definiri (238. §.).

515. Co-

515. Corollarium 7.

Si quaerantur exponentes differentiales ∂y, $\partial^2 y$ pro $\partial x = 1$ ex (508. §.); habebit ∂y valorem positivum, et $\partial^2 y$ valorem negativum, pro quavis abscissa $x = AP$: in parabola ergo crescentibus abscissis crescunt ordinatae (444. §.) (prouti id jam ex 509. §. elucet); et in quovis puncto M obvertit parabola concavitatem axi abscissarum AU (461. §.); radius vero curvedinis in quolibet puncto M continetur sequenti expressione (472. §.), fitque, pro $x = 0$, in vertice A aequalis semiparametro, adeoque aequalis subnormali (513. §.).

$$r = \frac{(p + 4x)^{\frac{3}{2}}}{2p^{\frac{1}{2}}}.$$

516. Corollarium 8.

Dato radio curvedinis $r = Mm$ (15. Fig.) pro indeterminato puncto M parabolae ZMX invenietur aequatio per (484. §.) ad ejus evolutam RmS (482. §.), quae, cum semper sit $ZR + Rm = r =$ expressioni algebraicae variablis x (515. §.), erit algebraice rectificabilis, ita ut quivis arcus futurus sit $Rm = r - ZR$. Fig. 15.

517. Corollarium 9.

Pro spatio parabolico $AMP = S$ (21. Fig.) erit $\partial S = \frac{2y^2 \partial y}{p}$ (508. 488. §.), hinc, cum pro $x = AP = 0$ debeat fieri $S = AMP = 0$, invenietur $S = AMP = \frac{2}{3p} y^3$ (247. 238. §.) $= \frac{2}{3} yx$ (508. §.): spatium $S = AMP$ aequatur ergo duabus tertiis rectanguli MdAP. Fig. 21.

518. Corollarium 10.

Si pro $x = Ba$, $y = aP$ (18. Fig.) arcus BP parabolae revolvatur circa axem BE; nascetur conoides parabolicum $BPp = S$, eritque ejus soliditas $S = \frac{1}{2}\pi p x^2$ (490. 508. §.): sed etiam superficies rotunda conoidis BPp per (508. 491. §.) ope calculi integralis facile determinabitur. Fig. 18.

519. Definitio.

Focus parabolae est punctum axeos, cui insistit ordinata semiparametro aequalis.

520. Co-

520. Corollarium 1.

Fig. 21. Si F (21. Fig.) supponatur esse focus parabolae, et ordinata $y = QF = \frac{1}{2}p$ pro $AF = E$ distantia foci a vertice A (519. §.); debebit esse $\frac{1}{4}p^2 = pE$ (508. §.), unde obtinetur $E = \frac{1}{4}p$, et $p = 4E$.

521. Corollarium 2.

In parabola dat $2E$ subnormalem (520. 513. §.), quadratum vero cujuslibet ordinatae est $y^2 = 4xE$ (520. 508. §.). Quamobrem, ducta ex foco F ad quodcunque parabolae punctum M recta FM, cum sit $FM = \sqrt{(MP^2 + FP^2)}$, fiet $FM = x + E = AP + AF$.

522. Corollarium 3.

Si praeterea ducatur recta Md parallela axi AU, quo fiat angulus $dMT = FTM$; erit, ob $FT = FM$ (521. 513. §.), etiam angulus $FMT = FTM = dMT$: tangens TM bissecat igitur angulum dMF.

523. Problema.

Invenire aequationem ad parabolam inter abscissas $u = Mn$ *in recta* ML *axi* AU *parallela a puncto* M, *cujus distantia* $MP = a$ *ab axe data sit, computatas, et ordinatas* $qn = z$ *tangenti* TM *parallelas.*

Solutio.

1. Pro ordinata orthogona $y = qq^1$ et abscissa $x = Aq^1$ erit $y = a + qr$, et $x = AP + u + nr$.

2. Est autem $AP = \frac{a^2}{p}$ (508. §.), $TP = \frac{2a^2}{p}$, $TM = \frac{a}{p}\sqrt{(p^2 + 4a^2)}$ per (512. §.); hinc, ob similitudinem triangulorum qnr, MTP, fiet $qr = \frac{pz}{\sqrt{(p^2 + 4a^2)}}$ et $nr = \frac{2az}{\sqrt{(p^2 + 4a^2)}}$.

3. Per (2) determinabitur y et x in (1), atque his valoribus in (508. §.) substitutis obtinebitur sequens aequatio.

$$z^2 - \left(\frac{p^2 + 4a^2}{p}\right)u = 0.$$

524. Corollarium.

Cuivis abscissae $u = Mn$ respondet duplex ordinata, una positiva $z = nq$, et altera negativa $z = np$, priori aequalis: recta ML axi AU parabolae parallela bissecat igitur omnem chordam qp tangenti ad M paral-

parallelam; consequenter omnis recta ML est una *diameter* parabolae, sicut ad omne punctum M una tangens TM potest duci (438 §.).

525. Definitio.

Curva, cujus naturam exprimit aequatio $y^2 = \frac{b^2 x}{a} - \frac{b^2 x^2}{a^2}$ inter coordinatas x, y, rectasque constantes a, b vocatur *Ellipsis*.

526. Corollarium 1.

Sumto in axe abscissarum VU (22. Fig.) puncto A pro origine abscissarum; erit in ellipsi ordinata $y = 0$ tam pro $x = 0$ quam pro abscissa positiva $x = a = AB$: cuivis autem abscissae negativae $x = -Ar$, et positivae $x = Ar' > AB = a$ respondebit ordinata y imaginaria: cuilibet demum abscissae positivae $x = AP < AB = a$ respondebit duplex ordinata, una positiva $y = MP$, et altera negativa $y = -Pm$ priori aequalis, utraque, abscissa $x = AP$ continuo crescente, primum crescens usque dum fiat $y = DC = CE = \frac{1}{2}b$ pro $x = AC = \frac{1}{2}AB = \frac{1}{2}a$, tum decrescens, abscissa crescente ultra AC, ita ut ordinatae $MP = mP$ et $Qp = qp$ aequidistantes a medio puncto C rectae AB inter se debeant esse aequales (525. §.). Fig. 22.

527. Corollarium 2.

Si ducatur chorda mq vel MQ parallela lineae abscissarum UV, et ex punctis m, q vel M, Q ducantur ordinatae mP, qp, vel MP, Qp, debebit esse $mP = qp$, et $MP = Qp$: ordinatae mP et qp, vel MP et Qp, jacent igitur in aequalibus distantiis a medio puncto C rectae AB (526. §.); consequenter recta DE in puncto C ad AB perpendicularis bissecat omnem chordam mq vel MQ in n.

528. Corollarium 3.

Et quaevis chorda mQ transiens per medium punctum C rectae AB bissecatur in C. Ductis enim ordinatis Qp, mP debebit esse $Cp = CP$, eoque ipso etiam $QC = mC$: secus enim esset e. gr. $CP > Cp$, adeoque aliqua $C\alpha = Cp$; hinc ordinata $\gamma\alpha = qp = Qp$ (526. §.), proinde angulus $\alpha C\gamma = QCp$, quod est impossibile, quin sit angulus $\alpha C\gamma = PCm$, punctumque α in P jaceat.

529. Corollarium 4.

Ellipsis continetur spatio finito ADBEA, habetque quatuor *Vertices* A, B, D, E, et duos *Axes* $AB = a$, $DE = b$ in *Centro* C sese bissecantes,

 quorum

quorum quivis totam ellipsim in duas partes congruentes, inter se similes et aequales dividit (526. 527. 528. 438. 439. §.): $AB = a$ vocant *axem principalem*, seu *transversum*, et $DE = b$ *axem secundarium* seu *conjugatum* (438. §.).

530. Corollarium 5.

Pro $Qp = y$, $Ap = x$, $Cp = u$ erit $x = \frac{1}{2}a + u$: pro hoc ergo valore loco x in (525. §.) obtinebitur aequatio $y^2 = \frac{1}{4}b^2 - \frac{b^2 u^2}{a^2}$ ad ellipsim inter ordinatas $y = Qp$ et abscissas $u = Cp$ a centro C computatas.

531. Corollarium 6.

In (525. §.) est $sy = \frac{b(a - 2x)sx}{2a\sqrt{(ax - x^2)}}$ positivi vel negativi valoris, prout est $x < \frac{1}{2}a$ vel $x > \frac{1}{2}a$, abscissis $x = AP$ vel $= Ap$ a vertice A computatis: tangens ad quodlibet punctum ellipseos cum subtangente cadet ergo ad eandem partem ordinatae, ad quam jacet origo A abscissarum, vel ad partem oppositam, prout punctum contactus in primo quadrante AD aut quarto EA, vel in secundo DB aut tertio BE jacuerit (500. §. 3. n.).

532. Corollarium 7.

Sic etiam ex (530. §.) obtinebitur $sy = -\frac{2busu}{a\sqrt{(a^2 - 4u^2)}}$ pro quavis abscissa u a centro C computata valoris negativi; unde per (500. §. 3. n.) sequitur, tangentem ad quodvis punctum ellipseos cum subtangente cadere ad partem ordinatae illi oppositam, ad quam jacet centrum C.

533. Corollarium 8.

Sumtis abscissis u a centro computatis, si T tangentem, sT subtangentem, N normalem, et sN subnormalem denotet, obtinebimus ex (532. 502. §.). sequentes expressiones:

$$T = \frac{y}{2bu}\sqrt{(a^4 - 4u^2(a^2 - b^2))};\quad sT = \frac{a^2 - 4u^2}{4u};$$

$$N = \frac{b}{2a^2}\sqrt{(a^4 - 4u^2(a^2 - b^2))};\quad sN = \frac{ub^2}{a^2}.$$

534. Corollarium 9.

Si tangens occurrat ordinatae in puncto contactus sub angulo m, et axi transverso AB sub angulo n; habebimus ob (532. §.) per (500. §. 4. n.) pro magnitudine absoluta horum angulorum sequentem formulam, quae eo deter-

determinato casu, quo fuerit $u=o$ vel $u=\frac{1}{2}a$ dabit $\text{Cos}\, m=o$ vel $\text{Sin}\, n=1$: nimirum tangens ad verticem axeos conjugati DE est huic axi perpendicularis, et parallela axi transverso AB; tangens vero ad verticem axeos transversi AB est eidem perpendicularis, et parallela axi conjugato DE.

$$\text{Cos}\, m=\text{Sin}\, n=\frac{2bu}{\sqrt{(a^4-4u^2(a^2-b^2))}}.$$

535. Corollarium 10.

Pro $\mu=DQ$ erit $\partial\mu=\frac{\partial u\sqrt{(a^4-4u^2(a^2-b^2))}}{\sqrt{(a^4-4a^2u^2)}}$ per (532. 469. §.); quodsi ergo integrale hujus exponentis per (359. 238. §.) ita determinetur, ut pro $u=o$ fiat quoque $\mu=o$; obtinebitur arcus $\mu=DQ$ debitus abscissae $u=Cp$; adeoque pro $u=CB=\frac{1}{2}a$ obtinebitur quadrans ellipseos DB.

536. Corollarium 11.

Et pro $S=DCpQ$ erit per (488. 530. §.) $\partial S=\frac{b}{2a}\partial u\sqrt{(a^2-4u^2)}$: hinc, quia debet fieri $S=DCpQ=o$ pro $u=Cp=o$, inveniemus per (289. 238. §.) $S=DCpQ=\frac{bu}{4a}\sqrt{(a^2-4u^2)}+\frac{ba}{8}\text{Arc Sin}\frac{2u}{a}$. Quamobrem pro $u=\frac{1}{2}a$, quo casu fit $\text{Arc Sin}\frac{2u}{a}=\text{Arc Sin}\,1=\frac{1}{2}\pi$, obtinebitur area quadrantis elliptici $DCB=\frac{1}{16}ab\pi$: area igitur totius ellipseos $ADBEA=\frac{1}{4}ab\pi$ est ad aream $\frac{a^2\pi}{4}$ circuli, qui super axe $AB=a$ transverso tanquam diametro descriptus cogitetur, sicut axis conjugatus b ad transversum a.

537. Corollarium 12.

Si pro $Ba=x$, $aP=y$ arcus ellipseos BP (18. Fig.) revolvatur circa axem transversum, generabit is segmentum $S=BPp$ sphaeroidis elliptici, cujus genesis revolutione totius ellipseos circa axem transversum absolvitur: igitur per (490. 525. §.) reperietur ∂S, unde integrando per (249. 238. §.) obtinebimus soliditatem segmenti $S=BPp=\frac{\pi b^2}{6a^2}(3ax^2-2x^3)$. Fig. 18.

Hinc pro $x=\frac{1}{2}a$ invenietur soliditas dimidii sphaeroidis elliptici $=\frac{\pi ab^2}{12}$, et soliditas integri sphaeroidis $=\frac{\pi ab^2}{6}$.

 538. Co-

538. Corollarium 13.

[illegible] §.) ope calculi integralis quaeri poterit superficies [illegible] segmenti, et dimidii, integrique sphaeroidis elliptici [illegible] §.)

539. Definitio.

Parameter axeos transversi a in ellipsi est tertia geometrice proportionalis $p=\frac{b^2}{a}$ post axem transversum a et conjugatum b: punctum vero in axe transverso a, cui insistit ordinata aequalis semiparametro $\frac{1}{2}p$ vocatur *Focus*, et hujus distantia a centro ellipseos *Excentricitas*, quam deinceps litera c designabit.

540. Corollarium 1.

Valor $b^2=ap$ (539. §.) substitutus in (525. 530. §.) dabit sequentes aequationes ad ellipsim inter ordinatam y, et abscissam x a vertice axeos transversi a, abscissamque u a centro computatam.

$$y^2=px-\frac{px^2}{a}. \qquad y^2=\frac{1}{4}ap-\frac{pu^2}{a}.$$

541. Corollarium 2.

Pro ordinata $y=\frac{1}{2}p$ erit abscissa, una positiva et altera negativa, $u=\pm\frac{1}{2}\sqrt{(a^2-b^2)}$ semper valoris realis, modo axis transversus a sit major axe conjugato b: in hac hypothesi habebit igitur ellipsis in axe
23. transverso AB (23. Fig.) duos focos F, f in aequalibus a centro C distantiis CF, Cf, eritque excentricitas $CF=Cf=c=\frac{1}{2}\sqrt{(a^2-b^2)}$ per (539. §.).

542. Corollarium 3.

Denotante $E=Bf=AF$ distantiam cujuslibet foci f, F a proximo vertice B, A, erit in ellipsi $4c^2=a^2-b^2$; $E=\frac{1}{2}a-c$; $p=a-\frac{4c^2}{a}$ $=4\left(E-\frac{E^2}{a}\right)$ per (541. 539. §.).

543. Corollarium 4.

Posita ordinata $MP=y$ pro abscissa $CP=u$, ductisque ad M ex focis f, F rectis fM, FM, erit $fM=\pm\sqrt{(y^2+(c+u)^2)}$, et $FM=\pm\sqrt{(y^2+(c-u)^2)}$: substitutis itaque valoribus loco y^2, c^2 ex (530. 542. §.) obti-

obtinebimus $fM = \pm\left(\frac{a}{2}+\frac{2cu}{a}\right)$ et $FM = \pm\left(\frac{a}{2}-\frac{2cu}{a}\right)$. Verum, quia pro $u = CA = \frac{1}{2}a$ debet fieri $fM = fA = c + \frac{1}{2}a$, et $FM = FA = \frac{1}{2}a - c$, perspicuum est, esse re ipsa $fM = \frac{a}{2}+\frac{2cu}{a}$ et $FM = \frac{a}{2}-\frac{2cu}{a}$.

544. Corollarium 5.

In ellipsi est summa rectarum $fM + FM$ ex focis ad quodvis punctum M ductarum aequalis axi $a = AB$ transverso (543. §.): eapropter est recta $fD = FD$ ex quovis foco ducta ad verticem axeos conjugati DE aequalis semiaxi transverso $AC = \frac{1}{2}a$.

545. Corollarium 6.

Ducta ad punctum M tangente TS, quae axi transverso AB in T occurrat, debebit esse per (533. §.)

$$TA = \frac{a^2 - 2au}{4u}. \quad TF = \frac{a^2 - 4cu}{4u}. \quad Tf = \frac{a^2 + 4cu}{4u}.$$

546. Corollarium 7.

Datis distantiis puncti interfectionis T tangentis TS et axeos transversi AB in ellipsi a focis f, F, dataque ordinata $MP = y$ ad punctum contactus, dabitur eo ipso tangens $TM = \frac{2MP}{b}\sqrt{TF \cdot Tf}$ per (545. 542. 533. §.).

547. Corollarium 8.

Ducta ex foco F parallela Fm rectae fM; erit $TF : Fm = Tf : fM = a : 2u$ (545. 543. §.); hinc fiet $Fm = TF \cdot \frac{2u}{a} = FM$ (545. 543. §.): et ideo erit quoque angulus $FmM = FMm$. Tangens cum rectis ex utroque foco ad punctum contactus ductis comprehendit igitur in ellipsi angulos aequales fMS et FMT; normalis MN idcirco bissecat angulum fMF (419 §.).

548. Corollarium 9.

Aequatio ad ellipsim pro abscissis $x = AP$ (22. Fig.) a vertice A computatis (540. §) dabit dy et d^2y per a, p, x, eritque d^2y pro quavis abscissa x valoris negativi: ellipsis obvertit nimirum in quovis puncto M Fig. 22.

 conca-

concavitatem axi transverso AB (461. §.); radius vero curvedinis in M comprehendetur ob (472. §.) sequenti formula, ex qua pro $x = 0$, et $x = a$ obtinetur $r = \frac{1}{2}p$ radius curvedinis in utroque vertice A, B axeos transversi AB, et, pro $x = \frac{1}{2}a$, $r = \frac{a^{\frac{3}{2}}}{2p^{\frac{1}{2}}} = \frac{1}{2}a\sqrt{\frac{a}{p}}$ radius curvedinis in utroque vertice D, E axeos conjugati.

$$r = \frac{1}{2p^2}\left(p^2 + \frac{4p}{a^2}(p-a)x^2 + \frac{4p}{a}(a-p)x\right)^{\frac{3}{2}}.$$

549. Corollarium 10.

Fig. 23. Pro recta $FM = z$ ex foco F ellipseos ad punctum M sub variabili angulo $\gamma = MFB$ ducta erit $z = \frac{1}{2}a - \frac{2cu}{a}$ (543. §.), et $u = CP = CF - FP = c - z\,Cos\gamma$, hinc et (542. §) $z = \frac{\frac{1}{2}b^2}{a - 2c\,Cos\gamma}$: quodsi ergo ponatur area sectoris $BMF = S$; erit $\partial S = \frac{\frac{1}{8}b^4\,\partial\gamma}{(a - 2c\,Cos\gamma)^2}$ per (489. §.), atque hic exponens differentialis per (333. §.) integratus, ita ut pro $\gamma = 0$ integrale evanescat, dabit aream BMF.

550. Definitio.

Curva, cujus naturam exprimit aequatio $y^2 = \frac{b^2x}{a} + \frac{b^2x^2}{a^2}$ inter coordinatas x, y et constantes rectas a, b vocatur *Hyperbola*.

551. Corollarium 1.

Fig. 24. Sumta origine abscissarum in puncto A rectae VU (24. Fig.) erit in hyperbola ordinata $y = 0$ tam pro $x = 0$ quam pro abscissa negativa $x = -AB = -a$: cuivis porro abscissae negativae $x = -Av$, existente $Av < AB = a$, respondebit ordinata y imaginaria; et cuilibet abscissae tam positivae $x = Ap$, quam negativae $x = -AP$, existente $AP > AB = a$, respondebit duplex ordinata y, una positiva, $y = Qp$ casu primo et $y = MP$ casu secundo, altera vero negativa priori aequalis, $y = -pq$ casu primo et $y = -Pm$ casu secundo, utraque in indefinitum crescens, abscissa continuo crescente, ita ut ordinatae $Qp = pq$ et $MP = Pm$ aequidistantes a punctis A, B inter se aequales sint (550. §.)

552. Co-

552. Corollarium 2.

Hinc (551. §.) eadem ratione, qua id pro ellipsi demonstravimus (527, 528. §.), elucet, omnem chordam MQ, mq perallelam lineae abscissarum VU a recta DE bissecari in n, si DE ad rectam AB in puncto medio C sit perpendicularis; omnemque chordam Qm transeuntem per C bissecari in C.

553. Corollarium 3.

Hyperbola extenditur per spatium indefinitum, habetque quatuor ramos AR, AS, Br, Bs, quorum bini priores in puncto A, bini vero posteriores in B concurrunt, tum illi et hi juxta lineam abscissarum AB in indefinitum excurrentes eo magis ab illa recedunt, quo magis producuntur (551. 437. §.). Habet praeterea hyperbola unum *Axem* AB$=$a definitae longitudinis, et duos *Vertices* A, B; alter autem *Axis* DE priorem in *Centro* C bissecans, atque ad hunc perpendicularis, in se indefinitae longitudinis (551. 552. 438. 439. §.), potest poni $=$b. Hoc modo habebimus in hyperbola, sicut in ellipsi, axem *principalem* seu *transversum* AB$=$a, et *secundarium* seu *conjugatum* DE$=$B.

554. Corollarium 4.

Axis transversus AB, utrinque productus, dividit spatium inter quamvis chordam Qq vel Mm ei perpendicularem arcumque QAq vel MBm interceptum in duas partes congruentes, ideoque inter se similes et aequales: chordae vero Qq, Mm aequidistantes a centro C et perpendiculares axi AB abscindunt segmenta QAq, MBm congruentia, inter se similia et aequalia (551. §.).

555. Corollarium 5.

Pro Qp$=$y, Ap$=$x, Cp$=$u erit $x = Cp - CA = u - \frac{1}{2}a$: hinc et ex (550. §.) obtinetur aequatio $y^2 = \frac{b^2 u^2}{a^2} - \frac{1}{4} b^2$ ad hyperbolam pro abscissis a centro C computatis.

556. Corollarium 6.

Prior aequatio (550. §.) pro abscissis x$=$Ap a vertice A computatis dat $\varepsilon y = \frac{b(a+2x)\varepsilon x}{2a\sqrt{(ax+x^2)}}$; et posterior (555. §.) pro abscissis u$=$Cp a centro C computatis dat $\varepsilon y = \frac{2bu\varepsilon u}{a\sqrt{(4u^2-a^2)}}$: utroque casu

est

est ergo sy pro omnibus abscissis, tam positivis, quam negativis, quibus reales ordinatae respondent (551. §.), valoris positivi; unde per (500. §. 5. n.) sequitur, tangentem cum subtangente ad quodlibet punctum hyperbolae ad eandem partem ordinatae cadere, ad quam jacet origo abscissarum sumta in vertice A, vel centro C.

557. Corollarium 7.

Sit T tangens ad quodcunque hyperbolae punctum ducta, sT subtangens, N normalis, sN subnormalis, y ordinata ad punctum contactus, u abscissa a centro computata, m angulus inter tangentem et ordinatam, et n angulus inter tangentem et axem transversum; erit per (556. 502. §.).

$$T = \frac{y}{2bu}\sqrt{(4u^2(b^2+a^2)-a^4)};\quad sT = \frac{4u^2-a^2}{4u};$$

$$N = \frac{b}{2a^2}\sqrt{(4u^2(b^2+a^2)-a^4)};\quad sN = \frac{ub^2}{a^2};$$

$$\mathrm{Sin}\,n = \mathrm{Cos}\,m = \frac{2bu}{\sqrt{(4u^2(b^2+a^2)-a^4)}}.$$

558. Corollarium 8.

Pro $u = \pm\frac{1}{2}a$ fiet (557. §.) $\mathrm{Sin}\,n = \frac{\pm ba}{\sqrt{b^2(\pm a)^2}} = \frac{\pm ba}{\pm ba} = 1$: tangens ergo in vertice hyperbolae A vel B est perpendicularis ad axem transversum AB.

559. Corollarium 9.

Posito hyperbolae arcu $\mu = AQ$, obtinebimus pro abscissis $u = Cp$ a centro computatis per (556. 469. §.) $s\mu = \frac{su\sqrt{(4u^2(a^2+b^2)-a^4)}}{\sqrt{(4a^2u^2-a^4)}}$: quodsi igitur hic exponens differentialis ita integretur, ut integrale $\mu = AQ$ pro $u = CA = \frac{1}{2}a$ fiat aequale nihilo, unde constantis pendet determinatio (238. §.); obtinebitur arcus $\mu = AQ$ cuivis abscissae $u = Cp$ debitus.

560. Corollarium 10.

Pro spatio autem $S = AQp$ habebimus per (555. 488. §.) $sS = \frac{b}{2a}su\sqrt{(4u^2-a^2)}$: invenietur igitur spatium AQp cuivis abscissae $u = Cp$ respondens, si sS per (289. 238. §.) sic integretur, ut pro $u = CA = \frac{1}{2}a$ fiat integrale $S = AQp = 0$.

561. Co-

561. Corollarium 11.

Arcus BP (18. Fig.) hyperbolae, respondens abscissae $x=Ba$, et or- Fig. 18.
dinatae $y=aP$, revolutus circa axem transversum BE (553. §.) generabit conoidem hyperbolicum PBp, cujus soliditas $S=\frac{\pi b^2}{6a^2}(3ax^2+2x^3)$ ope calculi integralis per (490. 550. 249. §.) facile determinatur: superficiem autem eius rotundam S per (491. 550. §.) pariter ope calculi integralis licebit determinare.

562. Corollarium 12.

Si ad vertices A, B hyperbolae (25. Fig.) ponantur perpendicula GH, gh bissecta ita, ut dimidium cujuslibet perpendiculi AG, AH, Bg, Bh aequetur semiaxi conjugato $\frac{1}{2}b$, tum per centrum C ducantur rectae Ll^1, L^1l; erit pro quavis abscissa $u=CP$, et ordinata $y=MP$ in m producta, $mP=\frac{bu}{a}$: igitur $mP^2-MP^2=\frac{1}{4}b^2$ (555. §.), et ideo $mM=\frac{b^2}{4(mP+MP)}$. Crescente itaque abscissa $u=CP$ in indefinitum, decrescet continuo mM, quin sit possibile, ut pro certa abscissa $u=CP$ fiat $mM=0$; quia secus fieret etiam $b^2=0$: rectae CL, Cl, CL^1, Cl^1 sunt ergo asymptoti ramorum AK, Ak, BK^1, Bk^1 (437. §.).

563. Definitio.

Parameter hyperbolae est tertia geometrice proportionalis $p=\frac{b^2}{a}$ post axem ejus transversum a et conjugatum b: punctum axeos transversi, in quo ordinata aequatur semiparametro, est *Focus*, et hujus distantia a centro hyperbolae *Excentricitas* c.

564. Corollarium 1.

Per (563. 550. 555. §.) obtinebimus sequentes aequationes ad hyper- Fig. 26.
bolam pro abscissis $x=AP$ (26. Fig.) a vertice A, et aliis $u=CP$ a centro C computatis.

$$y^2=px+\frac{px^2}{a}. \qquad y^2=\frac{pu^2}{a}-\frac{1}{4}ap.$$

565. Corollarium 2.

Posita ordinata $y=\frac{1}{2}p$ erit abscissa $u=\pm\frac{1}{2}\sqrt{(a^2+b^2)}$ per (555. §.), una positiva, et altera negativa, priori aequalis: hyperbola habet ergo in axe transverso AB utrinque producto duos focos F, f in aequalibus a

centro C distantiis CF, Cf, quo fit ut excentricitas sit $c = \frac{1}{2}\sqrt{(a^2+b^2)}$ per (563. §.).

566. Corollarium 3.

Sit $E = FA = fB$ distantia cujusvis foci hyperbolae a proximo vertice; erit $4c^2 = a^2+b^2$; $E = c - \frac{1}{2}a$; $p = \frac{4c^2}{a} - a = 4\left(E + \frac{E^2}{a}\right)$ per (565. 563. §.).

567. Corollarium 4.

Ductis ex utroque foco ad punctum M hyperbolae rectis fM, FM, erit $fM = \pm\sqrt{(y^2+(c+u)^2)}$ et $FM = \pm\sqrt{(y^2+(u-c)^2)}$: per (555. 566. §.) fiet ergo $fM = \pm\left(\frac{a}{2}+\frac{2cu}{a}\right)$ et $FM = \pm\left(\frac{a}{2}-\frac{2cu}{a}\right)$.. Cum autem pro $u = \frac{1}{2}a = CA$ debeat fieri $fM = fA = c + \frac{1}{2}a$, et $FM = FA = c - \frac{1}{2}a$; debet esse $fM = \frac{a}{2} + \frac{2cu}{a}$, et $FM = \frac{2cu}{a} - \frac{a}{2}$.

568. Corollarium 5.

Differentia rectarum $fM - MF$ ex focis hyperbolae ad quodvis ejus punctum M ductarum aequatur axi transverso $AB = a$.

569. Corollarium 6.

Si ad punctum M ducatur tangens TS axi transverso occurrens in T, petanturque distantiae hujus puncti a focis f, F, et vertice A, habebimus per (557. §.)

$$TA = \frac{2au - a^2}{4u};\quad TF = \frac{4cu - a^2}{4u};\quad Tf = \frac{a^2 + 4cu}{4u}.$$

570. Corollarium 7.

Datis autem distantiis TF, Tf, et ordinata $y = MP$ ad punctum contactus, dabitur tangens $TM = \frac{2y}{b}\sqrt{TF.Tf}$ (569. 557. §.).

571. Corollarium 8.

Ducatur ex foco F recta fm parallela alteri fM, erit TF : Fm = Tf : fM = a : 2u (569. 567. §.): erit itaque $Fm = TF.\frac{2u}{a} = FM$ [illegible]
proinde etiam angulus FmM = FMm. Tangens T[illegible]
M hyperbolae comprehendit cum rectis fM, FM
idem punctum ductis, angulos fMT, FMT
fM in R, normalis MN bissecat angulum [illegible]

572. Corollarium 9.

Aequatio ad hyperbolam pro abscissis x a vertice A computatis (564. §.) dabit exponentes differentiales $s y$, $\overset{2}{s} y$ per x, a, p, eritque $\overset{2}{s} y$ pro quavis abscissa valoris negativi: hyperbola obvertit idcirco axi transverso AB concavitatem in omnibus punctis (461. §.); radius autem curvaturae pro quovis hyperbolae puncto potest per (472. §.) sequenti formula exprimi, quae, posita abscissa $x = 0$ vel $x = -a = AB$, utroque casu dat $r = \frac{1}{2}p$ pro radio curvaturae in utroque vertice A, B.

$$r = \frac{1}{2p^2}\left(p^2 + \frac{4p}{a^2}(p+a)x^2 + \frac{4p}{a}(p+a)x\right)^{\frac{3}{2}}$$

573. Problema.

Dato Angulo $dCA =$ [illegible], *sub quo in ellipsi et hyperbola* (27. 28. Fig.) *chorda* cd *per centrum* C *transiens inclinatur ad axem transversum* AB, *invenire aequationem ad utramque curvam, inter abscissas* $Cm = v$ *in eadem chorda a centro computatas, et ordinatas* $z = nm$ *sub angulo* $nvC = \mu$ *ad axem transversum* AB *inclinatas.* Fig. 27. 28.

Solutio.

1. Ordinata orthogona ad punctum n sit $np = y$ respondens abscissae $Cp = u$; sit porro ms parallela ordinatae np, et mr parallela axi BA: erit

$y = rp + nr$ in Ellipsi et Hyperbola;

$u = ps - Cs$ in Ellipsi, et $u = ps + Cs$ in Hyperbola.

2. Quare, cum sit $rp = v \operatorname{Sin} \alpha$; $nr = z \operatorname{Sin} \mu$, $Cs = v \operatorname{Cos} \alpha$, $ps = z \operatorname{Cos} \mu$, habebimus

$y = z \operatorname{Sin} \mu + v \operatorname{Sin} \alpha$ in Ellipsi et Hyperbola;

$u = z \operatorname{Cos} \mu - v \operatorname{Cos} \alpha$ in Ellipsi;

$u = z \operatorname{Cos} \mu + v \operatorname{Cos} \alpha$ in Hyperbola.

3. Si jam hos valores substituas loco y, u in aequationibus ad ellipsim et hyperbolam (530. 555. §.), obtinebis sequentem aequationem, adhibitis signis superioribus pro ellipsi, et inferioribus pro hyperbola.

$$z^2 + \frac{2(a^2 \operatorname{Sin}\mu \operatorname{Sin}\alpha - b^2 \operatorname{Cos}\mu \operatorname{Cos}\alpha)vz}{a^2 \operatorname{Sin}\mu^2 \pm b^2 \operatorname{Cos}\mu^2}$$
$$+ \frac{4(a^2 \operatorname{Sin}\alpha^2 \pm b^2 \operatorname{Cos}\alpha^2)v^2 \mp a^2 b^2}{4(a^2 \operatorname{Sin}\mu^2 \pm b^2 \operatorname{Cos}\mu^2)} = 0.$$

 574. Co-

574. Corollarium 1.

Ducta ad d tangente Td ſupponatur ordinata $z = nm$ eſſe ei parallela; erit angulus $\mu = nvC = dTC$: quodſi ergo de ſit perpendicularis ad AB, fiet $\frac{de}{Te} = \frac{\text{Sin}\,\mu}{\text{Cof}\,\mu}$, $\frac{de}{Ce} = \frac{\text{Sin}\,\alpha}{\text{Cof}\,\alpha}$, hinc $\frac{de^2}{Te.Ce} = \frac{\text{Sin}\,\mu\,\text{Sin}\,\alpha}{\text{Cof}\,\mu\,\text{Cof}\,\alpha}$.. Quamobrem, cum pro $de = y$, $Ce = u$, ſubtangenteque Te per (530. 533. 556. 557. §.) debeat eſſe $\frac{de^2}{Te.Ce} = \frac{b^2}{a^2}$; erit in ellipſi et hyperbola $\frac{b^2}{a^2} = \frac{\text{Sin}\,\mu\,.\,\text{Sin}\,\alpha}{\text{Cof}\,\mu\,.\,\text{Cof}\,\alpha}$, et $b^2\,\text{Cof}\,\mu\,\text{Cof}\,\alpha = a^2\,\text{Sin}\,\mu\,\text{Sin}\,\alpha$.

575. Corollarium 2.

Pro ordinatis $z = nm$ tangenti Td parallelis abibit praecedens aequatio (573. §.), evanescente ſecundo termino ob (574. §.), in ſequentem. Haec autem aequatio dabit pro qualibet abſciſſa $v = \pm Cm$ poſitiva et negativa duplicem ordinatam z, unam poſitivam, et alteram negativam priori aequalem: quaevis ergo chorda principalis dc per centrum ellipſeos aut hyperbolae tranſiens eſt una *Diameter* ejusdem curvae, omnes chordas nN, n^1N^1 tangenti ad punctum d parallelas biſſecans (438. §.).

$$z^2 + \frac{4(a^2\,\text{Sin}\,\alpha^2 \pm b^2\,\text{Cof}\,\alpha^2)\,v^2 \mp a^2 b^2}{4(a^2\,\text{Sin}\,\mu^2 \pm b^2\,\text{Cof}\,\mu^2)} = 0.$$

576. Corollarium 3.

In ellipſi erit chorda ab *diameter conjugata* diametri de, ſi ab ſit tangenti Td parallela (575. 438. §.): in hyperbola nulla datur chorda per centrum C tranſiens tangenti Td parallela; licebit tamen rectam ab parallelam tangenti pro *diametro conjugata* diametri cd hyperbolae ſumere (575. 438. §.). Itaque in ellipſi et hyperbola erit diameter cd aequalis duplo valori abſciſſae v pro $z = 0$; diameter autem conjugata ab aequari debebit duplo valori ordinatae z pro $v = 0$: quare, denotante n diametrum cd, et m diametrum conjugatam ab; obtinebuntur ex (575. §.) ſequentes formulae pro iisdem diametris, ſignis ſuperioribus pro ellipſi, et inferioribus pro hyperbola adhibitis, nulla ratione habita imaginareitatis, qua diameter conjugata m hyperbolae afficitur, unde nihil aliud ſequitur, quam quod diameter conjugata nulli hyperbolae puncto occurrat, ſicut id poſito $u = 0$ in (555. §.) pro axe conjugato b aequatio ad hyperbolas ter y, u oſtendit.

$$m=\frac{ab}{\sqrt{(b^2\operatorname{Cof}\mu^2\pm a^2\operatorname{Sin}\mu^2)}}.\quad n=\frac{ab}{\sqrt{(b^2\operatorname{Cof}\alpha^2\pm a^2\operatorname{Sin}\alpha^2)}}.$$

577. Corollarium 4.

Dato angulo $\alpha = dCA$ inclinationis diametri dc ad axem transversum AB ellipſeos aut hyperbolae, datisque axibus a, b poterit determinari angulus $\mu = ACa$ inclinationis diametri conjugatae ab ad axem AB, et eo ipſo etiam angulus conjugationis $aCd = \alpha + \mu$ diametrorum cd, ab: erit enim ob (574. §.) in utraque curva

$$\operatorname{Tang}\mu = \frac{b^2}{a^2}\operatorname{Cot}\alpha;\ \operatorname{Sin}\mu = \frac{b^2\operatorname{Cof}\alpha}{\sqrt{(a^4\operatorname{Sin}\alpha^2+b^4\operatorname{Cof}\alpha^2)}};$$

$$\operatorname{Cof}\mu = \frac{a^2\operatorname{Sin}\alpha}{\sqrt{(a^4\operatorname{Sin}\alpha^2+b^4\operatorname{Cof}\alpha^2)}}.$$

578. Corollarium 5.

Hinc (576. 577. §.) naſcentur pro diametris m, n ellipſeos et hyperbolae ſequentes formulae I) II), et ex iis per (577. §.) obtinebuntur formulae III) IV).

$$\text{I. } m = b\sqrt{\frac{\operatorname{Cof}\alpha}{\operatorname{Sin}\mu\operatorname{Sin}(\alpha+\mu)}};\quad \text{II. } n = b\sqrt{\frac{\operatorname{Cof}\mu}{\operatorname{Sin}\alpha\operatorname{Sin}(\alpha+\mu)}};$$

$$\text{III. } mn = \frac{ab}{\operatorname{Sin}(\alpha+\mu)};\quad \text{IV. } ab = mn\operatorname{Sin}(\alpha+\mu).$$

579. Corollarium 6.

Si valores pro $\operatorname{Sin}\mu^2$ et $\operatorname{Cof}\mu^2$ ex (577. §.) in (576. §.) ſubſtituantur, tum pro ellipſi quaeratur ſumma, pro hyperbola vero differentia quadratorum diametrorum conjugatarum n^2, m^2; erit $n^2+m^2=a^2+b^2$ in ellipſi, et $n^2-m^2=a^2-b^2$ in hyperbola.

580. Definitio.

Binae curvae veterum, *Ciſſois* et *Conchois*, offerunt exempla curvarum algebraicarum tertii et quarti ordinis. Si ſuper data recta An (29. Fig.) tanquam diametro deſcribatur circulus AdntA, et ad punctum n ponatur tangens pq perpendicularis ad An, tum, abſciſſis quibuscunque partibus aequalibus na, nb in pq, ductisque ex A rectis Aa, Ab circuli peripheriam in c, d ſecantibus, fiat $AM=ca$, $Am=db$; curva, quae per hoc modo determinabilia tranſiverit, erit *Ciſſois*. Fig. 29.

581. Corollarium 1.

Pro coordinatis orthogonis $AP = x$, $MP = y$ erit in cissoide $AM^2 = ca^2 = y^2 + x^2$: cum igitur sit $ca^2 : an^2 = an^2 : Aa^2 = an^2 : (an^2 + An^2)$, et $an^2 : An^2 = MP^2 : AP^2$, proinde $an^2 = \frac{An^2 . MP^2}{AP^2}$; habebimus, pro radio r, et diametro $An = 2r$

$$(y^2 + x^2) : \frac{4r^2 y^2}{x^2} = \frac{4r^2 y^2}{x^2} : \left(\frac{4r^2 y^2}{x^2} + 4r^2\right);$$

hinc $x^2 (y^2 + x^2)^2 = 4 r^2 y^4$, et $xy^2 + x^3 = 2ry^2$.

Aequatio ad cissoidem erit itaque

$$x^3 + xy^2 - 2ry^2 = 0.$$

582. Corollarium 2.

Aequatio ad cissoidem dat $y^2 = \frac{x^3}{2r - x}$: cuivis ergo abscissae negativae $x = -Ae$ respondet ordinata y imaginaria: pro $x = 0$ autem, fiet etiam ordinata $y = 0$: cuilibet porro abscissae positivae $x = AP$ modo ea sit minor duplo radio $2r = An$, respondebit duplex ordinata, una positiva $y = MP$, et altera negativa $y = -mP$ priori aequalis, utraque continuo crescens, abscissa $x = AP$ indefinenter crescente, usque $x = An = 2r$, quo casu induit ordinata y valorem indefinite magnum.

583. Corollarium 3.

Quamobrem, si etiam in origine A abscissarum ponatur ad A n perpendicularis $p^I q^I$, continebitur tota cissois inter perpendiculares pq, $p^I q^I$ in indefinitum productas, constans duobus ramis AR, AS ad diversas partes lineae abscissarum An jacentibus, qui ex origine abscissarum A egredientes juxta illarum axem An et perpendiculum pq excurrent in indefinitum, ita ut ii, quo magis producti fuerint, eo magis sint recessuri a diametro An, eo magis vero accessuri ad perpendiculum pq (582. §.): pq est itaque asymptotus ramorum AR, AS.

584. Corollarium 4.

Aequatio ad cissoidem (581. §.) dabit pro y, ∂y, $\partial^2 y$, sequentes expressiones.

$$y = \sqrt{\frac{x^3}{2r - x}}; \quad \partial y = \frac{(3r - x)x^{\frac{1}{2}} \partial x}{(2r - x)^{\frac{3}{2}}}; \quad \partial^2 y = \frac{3r^2 \partial x^2}{x^{\frac{1}{2}}(2r - x)^{\frac{5}{2}}}.$$

Quare,

Quare, cum semper debeat esse $x < 2r$ (582. §.), patet, utrumque exponentem differentialem εy, $\overset{2}{\varepsilon} y$ pro quavis abscissa $x = AP$ habiturum valorem positivum; unde elucet, et ordinatas crescere, abscissis crescentibus, et cissoidem convexitatem in quovis puncto M, m obvertere axi abscissarum An (444. 461. §.).

585. Corollarium 5.

Per (584. 455. 456. §.) potest determinari tangens, subtangens, normalis, et subnormalis, situsque tangentis et subtangentis pro quovis puncto M et m: tangens cum subtangente, puta TM vel Tm cum TP, semper cadet ad partem ordinatae, ad quam jacet origo abscissarum A (584. 500. §. 3. n.); angulus vero $MTP = mTP$ dabitur per Tang T $= \frac{(3r - x)x^{\frac{1}{2}}}{(2r - x)^{\frac{3}{2}}}$ (500. §. 4. n.), fietque Tang $T = 0$ pro $x = AP = 0$: linea abscissarum An est ergo communis tangens ramorum AR, AS, et punctum A *Cuspis* cissoidis (584. 441. §.).

586. Corollarium 6.

Si petas radium curvedinis pro quocunque puncto M vel m cissoidis; invenies illum per (584. 472. §.) signo — adfectum, quod indicat, ipsum jacere in normali MN vel mN ad partem adversam tM vel tm, sic ut sit $tM = tm = \frac{rx^{\frac{1}{2}}(8r - 3x)^{\frac{3}{2}}}{3(2r - x)^2}$. Radius curvedinis ad quodvis punctum M cissoidis eo minor est, quo propius punctum M accedit ad cuspidem A; et ideo curvedo cissoidis crescit continuo versus cuspidem A (430. §.).

587. Corollarium 7.

Rectificatio cissoidis ope logarithmorum potest perfici. Si enim ponas $2r - x = z$; obtinebis ex (584. §.)

$$y = \sqrt{\frac{(2r - z)^3}{z}}; \quad \varepsilon y^2 = \frac{2r^3 + 3r^2 z - z^3}{z^3} \varepsilon z^2; \quad \varepsilon x = -\varepsilon z.$$

Pro quovis arcu $\mu = AM$ debebit ergo fieri per (469. §.)

$$\varepsilon \mu = \sqrt{\left(\frac{2r^3 + 3r^2 z - z^3}{z^3} \varepsilon z^2 + \varepsilon z^2\right)} = r \varepsilon z \, . \, \frac{\sqrt{(2r + 3z)}}{z^{\frac{3}{2}}}.$$

Integrale autem μ hujus exponentis differentialis licebit per (303. §.) perfecte determinare.

588. Co-

588. Corollarium 8.

Spatium vero $S = AMP$ respondens datae cuicunque abscissae $x = AP$ determinabile est per functionem algebraico-trigonometricam. Erit enim ob (584. §.) per (488. §.)

$$dS = \frac{x^{\frac{3}{2}} dx}{\sqrt{(2r - x)}}; \qquad S = \int \frac{x^{\frac{3}{2}} dx}{\sqrt{(2r - x)}}.$$

Quamobrem integrando per (302. §.) inveniemus

$$S = C - (\tfrac{1}{2} x^{\frac{3}{2}} + \tfrac{3}{2} r z^{\frac{1}{2}}) \sqrt{(2r - x)} + \frac{3r^2}{2} \text{ Arc Sin } \frac{x - r}{r}.$$

Verum, cum pro abscissa $x = 0$ debeat fieri $S = MAP = 0$, et Arc Sin $\frac{x - r}{r}$ = Arc Sin $-1 = -\frac{1}{2}\pi$; erit $C - \frac{1}{2}\pi . \frac{3r^2}{2} = 0$, hinc $C = \frac{3}{4}\pi r^2$, adeoque

$$S = \tfrac{3}{4}\pi r^2 - (\tfrac{1}{2} x + \tfrac{3}{2} r) \sqrt{(2rx - x^2)} + \frac{3r^2}{2} \text{ Arc Sin } \frac{x - r}{r}.$$

589. Corollarium 9.

Totum spatium intra An, AR, et np comprehensum, ramo cissoidis AR in indefinitum producto, obtinebitur ex (588. §.) pro $x = An = 2r$, nimirum $S = \frac{3}{2}\pi r^2$: hoc spatium est ergo triplum areae semicirculi Atn.

590. Definitio.

Fig. 30. 31. Si, datis duabus rectis EF, Ba (30. 31. 32. Fig.) sub angulo recto in D sese intersecantibus, capiantur in recta Ba partes $BD = DA = b$, $DC = a$ determinatae longitudinis, ut sit $b = a$ (31. Fig.), vel $b < a$ (30. Fig.) aut $b > a$ (32. Fig.), tum ex puncto C per quodvis punctum G rectae EF ducatur recta CM, fiatque $GM = GN = BD = DA = b$; jacebunt omnia hac ratione determinabilia puncta M, N in *Conchoide*. Punctum C vocatur *Polus* conchoidis; recta vero constans $b = BD = AD$ est ejus *Sagitta* seu *Parameter*, et EF *Axis* vel *Directrix*.

591. Corollarium 1.

Conchois constat duabus partibus RBS, UAV ad partes oppositas axis EF jacentibus, quarum communis asymptotus est ipse axis EF (437. §.).

592. Co-

592. Corollarium 2.

Si ex puncto M ad EF et Ba ducantur perpendicula MQ, MP, erit $MQ:MG = CP:CM = (CD+MQ):\surd(MP^2+CP^2)$, adeoque $MQ^2:MG^2 = (CD+MQ)^2:(DQ^2+(CD+MQ)^2)$: cum igitur sit $CD=a$, $MG=BD = DA=b$, si praeterea coordinatae orthogonae fiant $DQ=x$, $MQ=y$, habebimus pro aequatione ad conchoidem (590. §.)

$$y^2:b^2=(a+y)^2:(x^2+(a+y)^2);$$

$$y^4+2ay^3+(x^2+a^2-b^2)y^2-2ab^2y-a^2b^2=0;$$

$$(y+a)(y+a)(y+b)(y-b)+x^2y^2=0.$$

593. Corollarium 3.

Natura conchoidis, quae ope praecedentis aequationis potest explorari, pendet a relatione inter constantes rectas a, b (590. §.). Quidquid illae sint, aequales vel inaequales, deberet, pro $y=0$, fieri $a^2b^2=0$ (592. §.), quod est impossibile: impossibile ergo est, ut conchois axi EF in aliquo puncto occurrat, eumque tangat vel secet (591. §.).

594. Corollarium 4.

Quodsi autem ponatur $x=0$, nascetur aequatio $(y+a)(y+a)(y+b)(y-b)=0$ (592. §.) quatuor radicum realium $y=-a$, $y=-a$, $y=-b$, $y=b$ (170. 175. §.): conchois potest ergo sagittae (590. §.) quater occurrere omni casu; ter ex illa axeos EF parte, ad quam polus C (590. §.) jacet, nimirum bis in distantia $a=DC$, adeoque in ipso polo C, semel vero in distantia $b=DA$, et semel ex parte opposita axeos EF in distantia $b=DB$ (492. §. 4. n.).

595. Corollarium 5.

Omnis conchois habet punctum duplex in ipso polo C (594. 440. §.). Si est distantia poli ab axe major parametro, nimirum $a=CD>b=DA$ (30. Fig.); punctum duplex in polo C est conjugatum (498. §.). Si vero est $a=DC=b=DA$ (31. Fig.); punctum duplex in polo C (594. §.) coincidit cum puncto extimo A parametri DA, estque idcirco A punctum triplex (440. §.). Si denique est $b=DA>a=DC$; punctum duplex in polo C jacet in parametro DA inter ejus puncta extima D, A (32. Fig.).

596. Corollarium 6.

Ex aequatione ad conchoidem (592. §.), sumto exponente differentiali $\varepsilon x = 1$ pro constanti, nascentur sequentes aequationes differentiales per (141. 142. §.).

$$Z = y^4 + 2ay^3 + (x^2 + a^2 - b^2)y^2 - 2ab^2y - a^2b^2 = 0.$$

$$\varepsilon Z = (2y^3 + 3ay^2 + (x^2 + a^2 - b^2)y - ab^2)\varepsilon y + xy^2\varepsilon x = 0.$$

$$\overset{2}{\varepsilon} Z = \left\{ \begin{array}{l} (2y^3 + 3ay^2 + (x^2 + a^2 - b^2)y - ab^2)\overset{2}{\varepsilon} y \\ + (6y^2 + 6ay + x^2 + a^2 - b^2)\varepsilon y^2 \\ + 4xy\varepsilon y\varepsilon x + y^2\varepsilon x^2 \end{array} \right\} = 0.$$

597. Corollarium 7.

Prima aequatio differentialis (596. §.) dabit $\frac{\varepsilon y}{\varepsilon x} = 0$ pro $x = 0$ et $y = b = DB$ (30. 31. 32. Fig.), quidquid sit $a = DC$: omni ergo casu habebit conchois ad punctum B (594. §.) unam tangentem mn parallelam axi EF (503. §.).

598. Corollarium 8.

Sed etiam pro $x = 0$ et $y = -b = DA$, modo sit $DA > DC = a$, vel $DA < DC = a$ (32. 30. Fig.), obtinebitur $\frac{\varepsilon y}{\varepsilon x} = 0$ ex prima aequatione differentiali $\varepsilon Z = 0$ (596. §.): igitur etiam ad punctum A conchoidis (594. §.) dabitur una tangens pq parallela axi EF (503. §.), si parameter b sit major vel minor quam distantia a poli C ab axe EF.

599. Corollarium 9.

Verum pro $x = 0$ et $y = -a$ prodibit $\frac{\varepsilon y}{\varepsilon x} = \frac{0}{0}$ ex prima aequatione differentiali $\varepsilon Z = 0$ (596. §.), unde pro situ tangentis in puncto C seu polo conchoidis nihil potest colligi. Quamobrem sumatur in subsidium secunda aequatio differentialis $\overset{2}{\varepsilon} Z = 0$ (596. §.), quae pro $x = 0$ et $y = -a = DC$ dabit

$$\frac{\varepsilon x^2}{\varepsilon y^2} = \frac{b^2}{a^2} - 1, \text{ hinc } \frac{\varepsilon x}{\varepsilon y} = \pm\sqrt{\left(\frac{b^2}{a^2} - 1\right)}.$$

Quodsi ergo sit $b = DA < a = DC$ (30. Fig.); erit tangens in polo C tanquam puncto conjugato conchoidis (595. §.) imaginaria (504. §.). Si vero sit $b = DA = a = DC$ (31. Fig.); debebit tangens in puncto duplici C (505. §.) esse perpendicularis ad axem EF (504. §.), et C erit

Cuspis

Cuspis conchoidis (441. §.). Si demum sit $b = DA > a = DC$ (32. Fig.); extabunt ad C binae tangentes uv, rs (504. §.), et C erit *nodus* conchoidis (441. §.).

600. Definitio.

Exempla curvarum transcendentium (442. §.) desumemus a quadratrice, cycloide, et epicycloide. Si quadrans AB (33. Fig.) circuli radio $CB = r$ descripti, et radius AC dividantur in quibuscunque punctis E, P, sic ut sit $BE : BA = CP : CA$, tum, ducto radio CE, ex P ad AC ducatur perpendicularis PM eidem radio in M occurrens; jacebunt omnia puncta M hoc modo determinabilia in *Quadratrice dinostratis* AMD. Fig. 33.

601. Corollarium 1.

Pro arcu φ radio $= 1$ inter crura anguli BCE descripto et $z = CM$, erit $CP = z \operatorname{Sin} \varphi$, $BE = r\varphi$, $AB = \frac{1}{2} r\pi$: cum igitur sit $BE : BA = CP : CA$ (600. §.); erit $\varphi : \frac{1}{2}\pi = z \operatorname{Sin} \varphi : r$, hinc $z = \frac{2r\varphi}{\pi \operatorname{Sin} \varphi}$ aequatio ad quadratricem inter ordinatas $z = CM$ ad punctum C convergentes et angulos variabiles $\varphi = BCE$.

602. Corollarium 2.

Crescente angulo BCE, adeoque etiam arcu φ, crescet quoque $z = CM$, et pro $\varphi = \frac{1}{2}\pi$, anguloque $BCE = BCA = 90°$ fiet $z = r = AC$: quodsi autem decrescat angulus BCE, dum fiat tam is quam arcus $\varphi = 0$; fiet etiam $\operatorname{Sin} \varphi = 0$, et $z = CM = CD = \frac{2r}{\pi}$ (601. §.).

603. Corollarium 3.

Ex (601. §.) facile elicies aequationem inter coordinatas orthogonas $y = MP$, $x = CP$: erit enim

$$x = z \operatorname{Sin} \varphi = \frac{2r\varphi}{\pi};$$

$$y = z \operatorname{Cos} \varphi = \frac{2r\varphi \operatorname{Cos} \varphi}{\pi \operatorname{Sin} \varphi} = x \operatorname{Cot} \frac{\pi x}{2r}.$$

604. Corollarium 4.

Pro his coordinatis invenies in (603. §.) exponentem rationis differentialis functionis y

596. Corollarium 6.

Ex aequatione ad conchoidem (592. §.), fumto exponente differentiali $\varepsilon x = 1$ pro conftanti, nafcentur fequentes aequationes differentiales per (141. 142. §.).

$$Z = y^4 + 2ay^3 + (x^2 + a^2 - b^2)y^2 - 2ab^2y - a^2b^2 = 0.$$

$$\varepsilon Z = (2y^3 + 3ay^2 + (x^2 + a^2 - b^2)y - ab^2)\varepsilon y + xy^2\varepsilon x = 0.$$

$$\overset{2}{\varepsilon} Z = \left\{ \begin{array}{l} (2y^3 + 3ay^2 + (x^2 + a^2 - b^2)y - ab^2)\overset{2}{\varepsilon} y \\ + (6y^2 + 6ay + x^2 + a^2 - b^2)\varepsilon y^2 \\ + 4xy\varepsilon y\varepsilon x + y^2\varepsilon x^2 \end{array} \right\} = 0.$$

597. Corollarium 7.

Prima aequatio differentialis (596. §.) dabit $\frac{\varepsilon y}{\varepsilon x} = 0$ pro $x = 0$ et $y = b = DB$ (30. 31. 32. Fig.), quidquid fit $a = DC$: omni ergo cafu habebit conchois ad punctum B (594. §.) unam tangentem mn parallelam axi EF (503. §.).

598. Corollarium 8.

Sed etiam pro $x = 0$ et $y = -b = DA$, modo fit $DA > DC = a$, vel $DA < DC = a$ (32. 30. Fig.), obtinebitur $\frac{\varepsilon y}{\varepsilon x} = 0$ ex prima aequatione differentiali $\varepsilon Z = 0$ (596. §.): igitur etiam ad punctum A conchoidis (594. §.) dabitur una tangens pq parallela axi EF (503. §.), fi parameter b fit major vel minor quam diftantia a poli C ab axe EF.

599. Corollarium 9.

Verum pro $x = 0$ et $y = -a$ prodibit $\frac{\varepsilon y}{\varepsilon x} = \frac{0}{0}$ ex prima aequatione differentiali $\varepsilon Z = 0$ (596. §.), unde pro fitu tangentis in puncto C feu polo conchoidis nihil poteft colligi. Quamobrem fumatur in fubfidium fecunda aequatio differentialis $\overset{2}{\varepsilon} Z = 0$ (596. §.), quae pro $x = 0$ et $y = -a = DC$ dabit

$$\frac{\varepsilon x^2}{\varepsilon y^2} = \frac{b^2}{a^2} - 1, \text{ hinc } \frac{\varepsilon x}{\varepsilon y} = \pm\sqrt{\left(\frac{b^2}{a^2} - 1\right)}.$$

Quodfi ergo fit $b = DA < a = DC$ (30. Fig.); erit tangens in polo C tanquam puncto conjugato conchoidis (595. §.) imaginaria (504. §.
Si vero fit $b = DA = a = DC$ (31. Fig.); de
plici C (595. §.) effe perpendicularis ad axem E.

$$\varepsilon y = \frac{\text{Sin}\,\varphi\,\text{Cos}\,\varphi - \varphi}{\text{Sin}\,\varphi^2} \cdot \varepsilon x = \frac{\frac{1}{2}\text{Sin}\,2\varphi - \varphi}{\text{Sin}\,\varphi^2} \cdot \varepsilon x.$$

Hinc manifestum fit, ob (59. §.), εy semper habere valorem negativum: crescentibus abscissis $x = CP$ decrescunt ergo ordinatae $y = MP$ (444. §.), tangensque MT ad punctum M cum subtangente TP jacet ad partem ordinatae MP ei oppositam, ad quam jacet origo abscissarum C (500. §. 3. n.); et, sumto idcirco εy cum signis contrariis, obtinebis subtangentem cum angulis MTP, TMP per (500. §. 4. n.): nimirum

$$TP = \frac{r\varphi}{\pi} \cdot \frac{\text{Sin}\,2\varphi}{\varphi - \frac{1}{2}\text{Sin}\,2\varphi}.$$

$$\text{Tang}\,MTP = \frac{\varphi - \frac{1}{2}\text{Sin}\,2\varphi}{\text{Sin}\,\varphi^2}; \quad \text{Tang}\,TMP = \frac{\text{Sin}\,\varphi^2}{\varphi - \frac{1}{2}\text{Sin}\,2\varphi}.$$

Pro angulo $BCE = o$ debebit fieri $\varphi = o$, $\text{Sin}\,\varphi = o$, $\text{Sin}\,2\varphi = o$: igitur pro tangente in D erit $\text{Tang}\,TDP = o$, et subtangens $TP = TC = o$.

605. Definitio.

Fig. 34. Si diameter AB (34. Fig.) dati circuli perpendicularis sit ad rectam DE in B, et per quodcunque illius punctum P ducatur MM^1 parallela rectae DE peripheriam circuli in m, m^1 secans, sic ut sit arcus $Am = Am^1$, tum fiat $Am = mM = Am^1 = m^1M^1$; jacebunt omnia puncta M, M^1 hac ratione determinabilia in una curva, quae *Cycloïs* vulgaris seu *Trochoïs* vocatur.

606. Corollarium 1.

Pro radio $AC = r$ circuli *genitoris*, et $AP = x$, $MP = y$, erit $y = mP = Mm = mP + Am$ (605. §.). Est autem $mP = \sqrt{(2rx - x^2)}$, et tam mP sinus, quam $AP = x$ sinus versus arcus Am: quodsi ergo huic substituas arcum radio $= 1$ inter crura anguli ACm descriptum per (5. §. 2. Schol.), habebis sequentes aequationes ad cycloidem inter coordinatas orthogonas $x = AP$, $y = MP$.

$$y = \sqrt{(2rx - x^2)} + r\,\text{Arc Sin v}\,\frac{x}{r};$$

$$\text{vel } y = \sqrt{(2rx - x^2)} + r\,\text{Arc Sin}\,\frac{\sqrt{(2rx - x^2)}}{r}.$$

607. Corollarium 2.

Pro $x = AB = 2r$ erit $y = BE = r.\,\text{Arc Sin v}\,2 = r\pi =$ semiperipheriae AmB circuli genitoris (606. §.).

608. Co-

608. Corollarium 3.

Ordinata y cycloidis ſemper eſt imaginaria, tam pro abſciſſa negativa $x = -Ap$, quam pro poſitiva $x = Aq > AB = 2r$ (606. §.): tota cyclois continetur ergo inter parallelas DE, de per puncta extima diametri AB tranſeuntes.

609. Corollarium 4.

Differentiatio prima et ſecunda functionis y in (606. §.) dabit ſequentes exponentes differentiales:

$$\partial y = \partial x \sqrt{\frac{2r-x}{x}}; \quad \overset{2}{\partial} y = \frac{-r}{x\sqrt{(2rx - x^2)}}.$$

610. Corollarium 5.

Cum ſemper debeat eſſe $x < 2r$ (608. §.); ſemper erit ∂y valoris poſitivi, et $\overset{2}{\partial} y$ valoris negativi (609. §.): tangens MT cum ſubtangente TP pro quovis puncto M cycloidis jacebit igitur cum origine A abſciſſarum $x = AP$ ad eandem partem ordinatae mP (500. §. 3. n.); ipſa vero cyclois obvertet concavitatem axi abſciſſarum AB in quolibet puncto M (461. §.).

611. Corollarium 6.

Per (609. 500. §. 4. n.) invenientur pro ſitu tangentis reſpectu ordinatae MP et lineae abſciſſarum AB ſequentes expreſſiones:

$$\text{Tang } MTP = \sqrt{\frac{2r-x}{x}}; \quad \text{Tang } TMP = \sqrt{\frac{x}{2r-x}}.$$

Pro $x = AB = 2r$ erit tang $MTP = 0$; et pro $x = 0$ fiet tang $TMP = 0$: tangens in E vel A eſt igitur perpendicularis ad BE caſu primo, vel AB caſu ſecundo.

612. Corollarium 7.

Radius curvedinis in quolibet puncto M cycloidis erit ob (609. 472. §.) aequalis functioni $2\sqrt{(4r^2 - 2rx)} = 2.mB$, nimirum = duplae chordae mB: pro $x = AP = 0$ erit itaque radius curvedinis in puncto A aequalis quadruplo radio, ſeu duplae diametri circuli genitoris, nimirum $= 4r = 4.AC = 2.AB$.

613. Corollarium 8.

Pro arcu $\mu = AM$ per (469. 609. §.) invenies exponentem differentialem $\partial\mu = \partial x \sqrt{\frac{2r}{x}}$: ipse igitur arcus erit $\mu = AM = 2\sqrt{2rx}$. Hinc pro $x = 2r = AB$ debebit esse $AME = 2\sqrt{4r^2} = 4r$.

614. Corollarium 9.

Et pro spatio $S = AMP$ habebimus ob (606. 488. §.) exponentem differentialem.

$$ES = \partial x \sqrt{(2rx - x^2)} + r\partial x \text{ Arc Sin v} \frac{x}{r}.$$

Hinc si altera pars integretur per (257. §.), prodibit

$$S = rx \text{ Arc Sin v} \frac{x}{r} + \int \partial x \sqrt{(2rx - x^2} - \int \frac{rx\partial x}{\sqrt{(2rx - x^2)}}.$$

Adeoque per (289. 302. §.) debebit esse

$$S = AMP = \tfrac{1}{2}(r + x)\sqrt{(2rx - x^2)} + r(x - \tfrac{1}{2}r) \text{ Arc Sin v} \frac{x}{r}.$$

Quamobrem pro $x = 2r$ obtinebitur spatium $ABE = r(2r - \frac{1}{2}r)$ Arc Sin v $2 = \frac{3}{2}\pi r^2$ triplum areae AmB semicirculi genitoris.

615. Corollarium 10.

Ceterum facile ex (605. §.) colligitur, punctum A diametri AB percursurum cycloidem AME, si circulus genitor $AmBm^1A$ rotetur super recta DE.

616. Definitio.

Fig. 35. Si super peripheria EBF (35. Fig.) circuli radio CB descripti rotari cogitetur circulus $AbBaA$ cum priori jacens in eodem plano; progredietur punctum extimum A diametri AB in curva AA^1D, quam *Epicycloidem vulgarem* vocant.

617. Corollarium 1.

Si circulus genitor ab initiali situ $AbBa$ perveniat ad situm $A^1b^1B^1a^1$, ita ut diametri AB, ab situm A^1B^1, a^1b^1 obtineant; erit arcus $a^1B = aB = a^1B^1$: quodsi ergo ponatur arcus $a^1B = a^1B^1 = u$, et radius circuli genitoris $cB = c^1B^1 = r$, circuli vero, super quo ille rotatur, radius $CB = R$; erit angulus $a^1CB = \frac{u}{R}$, et angulus $a^1c^1B^1 = \frac{u}{r}$ (5. §. 3. Schol.).

618. Co-

618. Corollarium 2.

Ducta ad quodcunque punctum A^1 epicycloidis recta $CA^1 = v$, cum sit $A^1c^1 = r$, $Cc^1 = r+R$, et angulus $A^1c^1C = 180^\circ - B^1c^1C = 180^\circ - \frac{u}{r}$ (617. §.); habebimus per (5. §. 1. Schol. 36. Form.)

$$v^2 = r^2 + (r+R)^2 + 2r(r+R)\,\mathrm{Cof}\,\frac{u}{r}.$$

619. Corollarium 3.

Pro coordinatis orthogonis $x = CP$, $y = A^1P$ debebit ergo esse ob (618. §.)

$$y^2 = r^2 + (r+R)^2 + 2r(r+R)\,\mathrm{Cof}\,\frac{u}{r} - x^2.$$

620. Corollarium 4.

Ubi circulus genitor AbBa situm DMGN obtinuerit, puncto A in D, et B in G cadente; erit $u = BD = AaB = \pi r$: igitur $z^2 = CD^2 = r^2 + (r+R)^2 + 2r(r+R)\,\mathrm{Cof}\,\pi$ (618. §.); adeoque, ob $\mathrm{Cof}\,\pi = -1$, erit $z^2 = CD^2 = R^2$, et $z = R$.

621. Corollarium 5.

Ductis ex c^1 ad A^1P, CP perpendiculis c^1q, c^1p; erit angulus $A^1c^1q = a^1c^1B^1 + a^1CB = \frac{u}{r} + \frac{u}{R}$ (617. §.), et $x = CP = Cp + c^1q$, $y = A^1P = A^1q + c^1p$: cum ergo sit

$$Cp = Cc^1.\mathrm{Cof}\,a^1CB = (r+R)\,\mathrm{Cof}\,\frac{u}{R};$$

$$c^1q = A^1c^1.\,\mathrm{Cof}\,A^1c^1q = r\,\mathrm{Cof}\left(\frac{u}{r} + \frac{u}{R}\right);$$

$$A^1q = A^1c^1.\,\mathrm{Sin}\,A^1c^1q = r\,\mathrm{Sin}\left(\frac{u}{r} + \frac{u}{R}\right);$$

$$c^1p = Cc^1.\,\mathrm{Sin}\,a^1CB = (r+R)\,\mathrm{Sin}\,\frac{u}{R};$$

debebit esse

$$x = (r+R)\,\mathrm{Cof}\,\frac{u}{R} + r\,\mathrm{Cof}\,\frac{r+R}{R}\cdot\frac{u}{r};$$

$$y = r\,\mathrm{Sin}\,\frac{r+R}{R}\cdot\frac{u}{r} + (r+R)\,\mathrm{Sin}\,\frac{u}{R}.$$

622. Co-

622. Corollarium 6.

Epicyclois erit itaque curva algebraica, vel transcendens, prout radii r, R fuerint commensurabiles, vel incommensurabiles. Si enim radii r, R sint commensurabiles; dabuntur numeri integri k, l, pro quibus fiet $\frac{r}{R}=\frac{k}{l}$ hinc $R=\frac{rl}{k}$, et $\frac{r+R}{R}=\frac{k+l}{l}$: habebimus ergo in (621. §.)

$$x=(r+R)\,\mathrm{Cos}\,\frac{k}{l}\cdot\frac{u}{r}+r\,\mathrm{Cos}\,\frac{k+l}{l}\cdot\frac{u}{r}.$$

Quamobrem licebit utrumque cosinum in hac aequatione per $\mathrm{Cos}\frac{u}{r}$ exprimere, et si tum valor loco $\mathrm{Cos}\frac{u}{r}$ ex aequatione in (619. §.) determinabilis substituatur, determinabitur aequatio algebraica inter x, y, carens arcu u.

623. Problema.

Explorare situm tangentis ad quodvis punctum A' *epicycloidis.*

Solutio.

1. Pro $\frac{u}{R}=\varphi$, et $\frac{r+R}{Rr}u=\mu$, habebimus ob (621. §.) sequentes aequationes:

$$x=(r+R)\mathrm{Cos}\,\varphi+r\,\mathrm{Cos}\,\mu;$$
$$y=r\,\mathrm{Sin}\,\mu+(r+R)\,\mathrm{Sin}\,\varphi.$$

2. Si arcus u = Ba' crescat, decrescet x = CP, et crescet y = A'P, unde sequitur tangentem A'T cum subtangente TP cadere ad partem ordinatae A'P oppositam illi, ad quam jacet origo abscissarum C (500. §. 3. n.).

3. Erit autem in (1) $\varepsilon x=(r+R)\varepsilon\,\mathrm{Cos}\,\varphi+r\,\varepsilon\,\mathrm{Cos}\,\mu$, et $\varepsilon y=r\,\varepsilon\,\mathrm{Sin}\,\mu+(r+R)\varepsilon\,\mathrm{Sin}\,\varphi$, seu $\varepsilon x=-(r+R)\,\mathrm{Sin}\,\varphi\,\varepsilon\varphi-r\,\mathrm{Sin}\,\mu\,\varepsilon\mu$, et $\varepsilon y=r\,\mathrm{Cos}\,\mu\,\varepsilon\mu+(r+R)\,\mathrm{Cos}\,\varphi\,\varepsilon\varphi$: quodsi ergo juxta (1) exponentes differentiales $\varepsilon\varphi$, $\varepsilon\mu$ determinentur per εu, valorque pro εx sumatur cum signis contrariis; debebit esse per (500. §. 4. n.)

$$\mathrm{Tang}\,PA'T=\frac{\varepsilon x}{\varepsilon y}=\frac{\mathrm{Sin}\,\varphi+\mathrm{Sin}\,\mu}{\mathrm{Cos}\,\varphi+\mathrm{Cos}\,\mu}=\mathrm{Tang}\,\tfrac{1}{2}(\varphi+\mu).$$

Et

Et ob $\varphi + \mu = \frac{2r+R}{rR} \cdot u$ erit

$$\text{Tang } PA^1T = \text{Tang } \tfrac{1}{2} \cdot \frac{2r+R}{rR} \cdot u.$$

624. Corollarium 1.

Pro situ tangentis respectu lineae abscissarum, et normalis A^1S respectu ordinatae erit ob (623. §.)

$$\text{Cot } A^1TC = \text{Cot } PA^1S = \frac{\text{Sin } \varphi + \text{Sin } \mu}{\text{Cos} \varphi + \text{Cos} \mu} =$$

$$= \text{Tang } \tfrac{1}{2}(\varphi + \mu) = \text{Tang } \tfrac{1}{2} \cdot \frac{2r+R}{rR} \cdot u.$$

625. Corollarium 2.

Ducta chorda A^1a^1, erit angulus $a^1 A^1 c^1 = \tfrac{1}{2} A^1 c^1 b^1 = \tfrac{1}{2} a^1 c^1 B^1 = \frac{u}{2r}$ (617. §.); ob $A^1c^1q = \frac{u}{r} + \frac{u}{R}$ (621. §.) autem, erit $c^1A^1q = 90° - \left(\frac{u}{r} + \frac{u}{R}\right)$: habebimus ergo angulum $a^1A^1q = a^1A^1c^1 + c^1A^1q = 90° - \tfrac{1}{2}\left(\frac{2r+R}{rR}\right)u$. Debet ergo esse Cot $a^1A^1q = \text{Tang } \tfrac{1}{2} \cdot \frac{2r+R}{rR} \cdot u$. Angulus intra chordam A^1a^1 et ordinatam A^1P aequatur itaque angulo intra normalem A^1S et ordinatam A^1P (624. §.), et ideo debet normalis A^1S cum chorda A^1a^1 coincidere, et tangens A^1T esse perpendicularis ad chordam A^1a^1.

626. Problema.

Rectificare arcum epicycloidis $\alpha = AA^1$.

Solutio.

Determinatis juxta (623. §. 1. 3. n.) valoribus pro ∂x, ∂y, obtinebimus

$$\partial y^2 + \partial x^2 = \frac{(r+R)^2}{R^2} \partial u^2 (2(1 + \text{Sin } \varphi \text{ Sin } \mu + \text{Cos} \varphi \text{ Cos} \mu))$$

$$= \frac{(r+R)^2}{R^2} \partial u^2 . 2 (1 + \text{Cos} (\mu - \varphi))$$

$$= \frac{(r+R)^2}{R^2} \partial u^2 . 2 \left(1 + \text{Cos} \frac{u}{r}\right) .$$

Per (469. §.) erit ergo

$$s\alpha = \frac{r+R}{R}\, su \,.\, 2\sqrt{\frac{1+\mathrm{Cos}\,\frac{u}{r}}{2}},$$

$$\text{seu } s\alpha = 2\,\frac{r+R}{R}\, su\,\mathrm{Cos}\,\tfrac{1}{2}\cdot\frac{u}{r}\,.$$

$$= \frac{4r(r+R)}{R}\, s\,\mathrm{Sin}\,\tfrac{1}{2}\cdot\frac{u}{r}\,:$$

Est ergo arcus $AA^{1} = \alpha = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\frac{u}{2r} + C$: cumque pro arcu $u = Ba^{1} = o$ debeat fieri etiam arcus $\alpha = AA^{1} = o$, et $\mathrm{Sin}\,\frac{u}{2r} = o$; debet esse constans $C = o$, hinc

$$\alpha = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\frac{u}{2r}\,.$$

627. Corollarium.

Pro arcu $u = BD = AaB = r\pi$, erit $\alpha = AA^{1}D$: igitur per (626. §.)

$$AA^{1}D = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\tfrac{1}{2}\pi = \frac{4r(r+R)}{R}\,.$$

628. Problema.

Invenire radium curvaturae pro quovis puncto A' epicycloidis.

Solutio.

Per (623. §. 1. 2. n.) et (626. §.) determinabis radium curvaturae pro puncto A' sequenti calculo.

$$\varphi = \frac{u}{R};\quad \mu = \frac{r+R}{Rr}\,u;$$

$$sx = \frac{r+R}{R}\, su\,(\mathrm{Sin}\,\varphi + \mathrm{Sin}\,\mu);$$

$$sy = \frac{r+R}{R}\, su\,(\mathrm{Cos}\,\varphi + \mathrm{Cos}\,\mu);$$

$s^{2}y$

$$\dot{s}y^2+\dot{s}x^2=\frac{(r+R)^2}{R^2}\,su^2.2(1+\mathrm{Cos}(\varphi-\mu));$$

$$\overset{2}{s}x=\frac{r+R}{rR^2}\,su^2(r\,\mathrm{Cos}\,\varphi+(r+R)\mathrm{Cos}\,\mu);$$

$$\overset{2}{s}y=-\frac{r+R}{rR^2}\,su^2(r\,\mathrm{Sin}\,\varphi+(r+R)\mathrm{Sin}\,\mu);$$

$$\overset{2}{s}y\,sx-\overset{2}{s}x\,sy=$$

$$=\frac{(2r+R)(r+R)^2}{rR^2}\,su^3(1+\mathrm{Sin}\,\varphi\,\mathrm{Sin}\,\mu+\mathrm{Cos}\,\varphi\,\mathrm{Cos}\,\mu)$$

$$=\frac{(2r+R)(r+R)^2}{rR^2}\,su^3(1+\mathrm{Cos}(\mu-\varphi)).$$

Per (474 §.), si radius curvaturae in puncto A' vocetur Z, erit ergo

$$Z=\frac{\left(\frac{(r+R)^2su^2}{R^2}\cdot 2(1+\mathrm{Cos}(\mu-\varphi))\right)^{\frac{3}{2}}}{\frac{(2r+R)(r+R)^2}{rR^2}\,su^3(1+\mathrm{Cos}(\mu-\varphi))}=$$

$$=\frac{4r(r+R)}{2r+R}\sqrt{\frac{1+\mathrm{Cos}(\mu-\varphi)}{2}}.$$

Consequenter est

$$Z=\frac{4r(r+R)}{2r+R}\sqrt{\frac{1+\mathrm{Cos}\frac{u}{r}}{2}}=\frac{4r(r+R)}{2r+R}\,\mathrm{Cos}\frac{u}{2r}.$$

629. Corollarium 1.

Radium curvedinis pro puncto A obtinebis ex (628. §.), posito arcu $u=Ba'=0$, quo casu fiet $\mathrm{Cos}\frac{u}{2r}=\mathrm{Cos}\,0=1$; igitur radius curvaturae in A fiet $Z=\frac{4r(r+R)}{2r+R}$.

 630. Co-

Per (469. §.) erit ergo

$$s\alpha = \frac{r+R}{R}\, s u \,.\, 2\sqrt{\frac{1+\mathrm{Cos}\,\frac{u}{r}}{2}},$$

$$\text{seu } s\alpha = 2.\frac{r+R}{R}\, s u\, \mathrm{Cos}\,\tfrac{1}{2}\cdot\frac{u}{r}.$$

$$= \frac{4r(r+R)}{R}\, s\, \mathrm{Sin}\,\tfrac{1}{2}\cdot\frac{u}{r}.$$

Est ergo arcus $A A^{2} = \alpha = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\frac{u}{2r} + C$: cumque pro arcu $u = B a^{1} = 0$ debeat fieri etiam arcus $\alpha = A A^{2} = 0$, et $\mathrm{Sin}\,\frac{u}{2r} = 0$; debet esse constans $C = 0$, hinc

$$\alpha = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\frac{u}{2r}.$$

627. Corollarium.

Pro arcu $u = BD = A a B = r\pi$, erit $\alpha = A A^{2} D$: igitur per (626. §.)

$$A A^{2} D = \frac{4r(r+R)}{R}\,\mathrm{Sin}\,\tfrac{1}{2}\pi = \frac{4r(r+R)}{R}.$$

628. Problema.

Invenire radium curvaturae pro quovis puncto A' epicycloidis.

Solutio.

Per (623. §. 1. 2. n.) et (626. §.) determinabis radium curvaturae pro puncto A' sequenti calculo.

$$\varphi = \frac{u}{R};\quad \mu = \frac{r+R}{Rr}\,u;$$

$$s x = \frac{r+R}{R}\, s u\,(\mathrm{Sin}\,\varphi + \mathrm{Sin}\,\mu);$$

$$s y = \frac{r+R}{R}\, s u\,(\mathrm{Cos}\,\varphi + \mathrm{Cos}\,\mu);$$

$$\partial y^2 + \partial x^2 = \frac{(r+R)^2}{R^2}\,\partial u^2 . 2(1+\mathrm{Cos}(\varphi-\mu));$$

$$\overset{2}{\partial} x = \frac{r+R}{rR^2}\,\partial u^2 (r\,\mathrm{Cos}\,\varphi + (r+R)\mathrm{Cos}\,\mu);$$

$$\overset{2}{\partial} y = -\frac{r+R}{rR^2}\,\partial u^2 (r\,\mathrm{Sin}\,\varphi + (r+R)\,\mathrm{Sin}\,\mu);$$

$$\overset{2}{\partial} y\,\partial x - \overset{2}{\partial} y\,\partial x =$$

$$= \frac{(2r+R)(r+R)^2}{rR^2}\,\partial u^3 (1 + \mathrm{Sin}\,\varphi\,\mathrm{Sin}\,\mu + \mathrm{Cos}\,\varphi\,\mathrm{Cos}\,\mu)$$

$$= \frac{(2r+R)(r+R)^2}{rR^2}\,\partial u^3 (1 + \mathrm{Cos}(\mu-\varphi)).$$

Per (474. §.), si radius curvaturae in puncto A' vocetur Z, erit ergo

$$Z = \frac{\left(\frac{(r+R)^2\,\partial u^2}{R^2} \cdot 2(1+\mathrm{Cos}(\mu-\varphi))\right)^{\frac{3}{2}}}{\frac{(2r+R)(r+R)^2}{rR^2}\,\partial u^3 (1+\mathrm{Cos}(\mu-\varphi))} =$$

$$= \frac{4r(r+R)}{2r+R}\sqrt{\frac{1+\mathrm{Cos}(\mu-\varphi)}{2}}.$$

Consequenter est

$$Z = \frac{4r(r+R)}{2r+R}\sqrt{\frac{1+\mathrm{Cos}\frac{u}{r}}{2}} = \frac{4r(r+R)}{2r+R}\,\mathrm{Cos}\frac{u}{2r}.$$

629. Corollarium 1.

Radium curvedinis pro puncto A obtinebis ex (628. §.), posito arcu $u = Ba' = 0$, quo casu fiet $\mathrm{Cos}\frac{u}{2r} = \mathrm{Cos}\,0 = 1$; i[illegible] radius curvaturae in A fiet $Z = \frac{4r(r+R)}{2r+R}$.

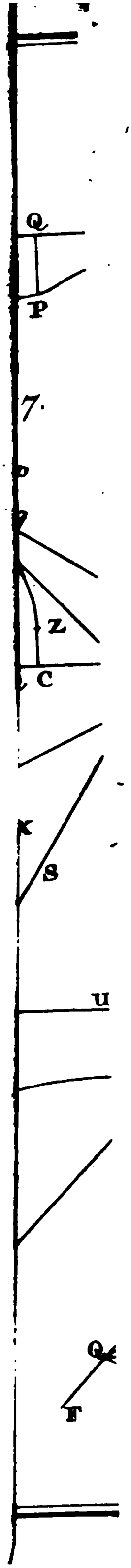
630. Co-

630. Corollarium 2.

Radium vero curvaturae pro puncto D invenies ex (628. §.) ponendo arcum $u = BD = AaB = r\pi$, adeoque $\mathrm{Cof}\frac{u}{2r} = \mathrm{Cof}\frac{1}{2}\pi = 0$: quare radius curvaturae in puncto D est $Z = 0$.

Error corrigendus.

In 301. §. pag. 130. debet pars prima feriei habere factorem $2\sqrt{(\alpha+\beta x)^3}$ loco $2\sqrt{\alpha+\beta x}$.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